Projectile Motion in Two Dimensions: With Air Resistance Considered

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January 13, 2008

Abstract

High school physics teacher makes a pet project of programing his calculator to calculate ballistic drop for his .17 Cal HMR rifle considering altitude, temperature, humidity, and a number of other local variables. Many different techniques and approaches are tried in order to reduce the processing time of the drop program. The scientific calculator is convenient to use on the field. However, it is very limited in the scope of its processing power.

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Initially, I tried separating the forces into X and Y coordinates, but I found that this is not directly possible because the force of friction is proportional to the square of the velocity. Since the X and Y components are not related as a ratio as they are in linear systems of vectors, it is not possible to separate the variables in a straight foreword manor. This was the greatest hurdle that I had to overcome. As a habit of working with simpler systems, I went straight for separating the variables into X and Y, and arrived at this incorrect result ($F_x = -kv_x^2 = ma_x$) where k is constant over a small range of velocities ($k = \frac{\rho C_d A}{2m}$). That haven been discovered, I was able to proceed and correctly predict bullet trajectories by using the correct equation $\vec{a} = -k|\vec{v}|\vec{v} - g\hat{y}$. This equation reveals that Y and X are inseparable. I had initially assumed that Y was small and negligible, but after careful consideration I had realized that even though the Y was so small in comparison, its presence tilts a very large friction force vector away from the horizontal axis. Therefore the Y component of the friction force is not negligible even though the Y component velocity is so small in comparison to the X component. However, this is small ratio of X to Y velocity only holds true for flat fire scenarios (trajectories that have a very small take off angle).

I will start out by talking about the sum of the forces in the X direction before the bullet exits the muzzle. As soon as the bullet leaves the barrel, we are concerned with external ballistics. At this point the term F_p

(force on the bullet due to the powder burning) drops to zero.

$$\sum \vec{F} = \vec{F}_p - \vec{F}_d = m\vec{a} \qquad (1)$$

Where

$$\vec{F_p} = 0$$

Eq. 1 Gives:

$$-\vec{F_d} = m\vec{a} \tag{2}$$

In typical Internet nerd fashion, I go looking to the Wikipedia for answers. There, I find that:

$$F_d = \frac{1}{2}\rho v^2 C_d A \tag{3}$$

In order to convert the drag term into a vector:

$$\vec{F_d} = \frac{1}{2}\rho C_d A|\vec{v}|\vec{v} \tag{4}$$

Where ρ is the density of the fluid (air in our case), A is the cross sectional area of the projectile($A = \pi r^2$), and C_d is the coefficient of drag.

from 2 and 3, and dividing both sides by m, I get:

$$\vec{a} = \frac{-\rho C_d A|\vec{v}|\vec{v}}{2m} \tag{5}$$

This is a differential equation, that cannot be solved directly by integration because of the inseparability of X and Y. Moreover, the coefficient of drag is not constant over the range of velocities for the projectile. This function is not well understood, and quite complex. There are only numeric empiricle functions for this relationship. Therefore, the only possible solution is through a numeric means.

We will find that it quite convenient to include one of the velocity terms into the coefficient function. This allows us to separate the vector into X and Y components and allowing the differential equation to be solved in a much simpler manor as I will show bellow. This is done by creating a numeric function known as the G function. It is based on the coefficient of drag for a standard Krupp artillery projectile with a diameter of 1 inch and a weight of 1 pound. It was found out in the late 1800's that it was possible to measure the drag coefficient for a standard bullet, and using a form factor, this drag function could be extrapolated using this form factor to work for all different kinds of projectiles. The "G"' function is given by:

$$G(M) = \frac{\pi}{8} C_{d_{G1}}(M)\rho_o|\vec{v}| \tag{6}$$

At this point it is useful to define the ballistic coefficient. The ballistic coefficient commonly used to quantify the ballistic properties of projectiles by most all of the modern ammunition manufacturers. One of the conveniences of the ballistic coefficient is that for any given projectile (regardless of its mass and sectional density) it will fly along the same path predicted by our differential equation. It is defined as:

$$BC = \frac{w}{id^2} \tag{7}$$

It is assumed that $C_d(M)$ changes proportionally to $C_{d_{G1}}(M)$ Therefore, i is the "constant" form factor of the bullet given by:

$$i = \frac{C_d(M)}{C_{d_{G1}}(M)} \tag{8}$$

and: w=weight of projectile in pounds, d=diameter of projectile in inches, and $C_{d_{G1}}(M)$=the coefficient of drag for the standard Krupp's bullet as a function of the Mach number of that bullet ($M = \frac{V_{bullet}}{V_{sound}}$).

Rearranging Eq. 6 we find that:

$$BC = \frac{C_{d_{G1}} w}{C_d d^2} \tag{9}$$

If $A = \pi r^2$, then $A = \frac{1}{4}\pi d^2$, and $d^2 = \frac{4}{\pi}A$. Substituting this result into Eq. 8 yields:

$$BC = \frac{\pi C_{d_{G1}} w}{4 C_d A} \tag{10}$$

This is a rather ugly result, but when the G function is divided by this result we find that:

$$\frac{G}{BC} = \frac{\rho C_d A}{2w}|\vec{v}| \tag{11}$$

This is almost the same factor as our original constant k with an additional $|\vec{v}|$ term, and instead of mass, we have weight. It turns out that the mass is referring to the bullet (measured in grains), and the weight in pounds is used for the dimensions of BC also referring to the mass/weight of the bullet. $1pound = 7000grains$, so if we are careful to convert the bullet mass into pounds before we do any calculations, we find that the ρ (air density) in the G function is in units of $\frac{lb}{ft^3}$, and therefore cancels out quite nicely – even though these crazy English seem to use mass and weight interchangeably.

In the beginning of our discussion I wrote the differential equation as $\vec{a} = -k|\vec{v}|\vec{v} - g\hat{y}$ where $k = \frac{\rho C_d A}{2m}$.

$$k|\vec{v}| = \frac{G}{BC} \tag{12}$$

We can redefine k using the G function and the ballistic coefficient. Absorbing the $|\vec{v}|$ term into k just like the G function. We call this new constant $k\prime$. Therefore:

$$k\prime = k|\vec{v}| = \frac{G}{BC} \tag{13}$$

As we shall see later, this is a very convenient because this constant shows up all over the place. With this simplification we can now proceed to break up the vector equations (1) into X and Y components. For the acceleration in the X direction we get:

$$a_x = -k\prime v \cos\theta \tag{14}$$

Where θ is the angle the projectile is traveling with respect to the horizontal at any given instant. Since $\cos\theta = \frac{v_x}{v}$.

$$a_x = -k\prime v_x \tag{15}$$

For small changes in velocities we can treat G is constant. Therefore:

$$-\frac{1}{k\prime}\int_{v_{x_o}}^{v_x} \frac{1}{v_x} dv_x = \int_0^t dt \tag{16}$$

Integrating gives:

$$\frac{1}{k\prime} ln(\frac{v_{x_o}}{v_x}) = t \tag{17}$$

Solving for v_x yields:

$$v_x = v_{x_o} e^{(-k\prime t)} \tag{18}$$

This is all we need to find how the velocity is affected by time. This equation can be used with an iterator changing the G in small steps so that it imitates a smooth function. These are useful equations, but it turns out that it is more valuable to determine the change in velocity as a function of distance so the distance the bullet travels in each segment of the range can be plugged in to find the velocity as a function of range, not time. Substitution of variables can be used so the distance the bullet travels in each segment of the range can be plugged in to find the velocity as a function of range, not time. Substitution of variables can be used.

$$a_x \frac{dx}{dx} = \frac{dv_x}{dx}\frac{dx}{dt} = \frac{dv_x}{dx} v_x = -k\prime v_x \tag{19}$$

Cross canceling v_x yields:

$$\frac{dv_x}{dx} = -k\prime \tag{20}$$

This is a very simple and elegant solution that when integrated gives the result:

$$v_x - v_{x_o} = -k\prime x \tag{21}$$

Plugging values into Eq. 21 and substituting the velocities back into Eq. 17 will give us the time of flight for the projectile. This in turn can be used to determine the drop of the bullet.

Now let us consider the differential equation in the y direction:

$$a_y = -k\prime v_y - g \tag{22}$$

Then:

$$\int_{v_{y_o}}^{v_y} \frac{1}{(k\prime v_y + g)} dv_y = \int_0^t dt \tag{23}$$

Solving for v_y we get the nasty but useful result:

$$v_y = \frac{1}{k\prime}[e^{-k\prime t}(k\prime v_{y_o} + g) - g] \tag{24}$$

This equation can be used with the small time of flight interval in an iterator that adjust for G the same as in the x differential equation. Note that G is a function of v not v_y. I ran this out to 400 yards and received good data (under 1 inch of error as compared to two known reliable ballistics programs). The next obstacle to overcome is finding ways to do this calculation more efficiently. The next step we shall take is to find y as a function of t as this would be the drop of the bullet. This can be found by integrating the last equation.

$$\int_{y_o}^{y} dy = \int_{0}^{t} \frac{1}{k\prime}[e^{-k\prime t}(k\prime v_{y_o} + g) - g]dt \tag{25}$$

This produces:

$$y = \frac{1}{k\prime}(v_{y_o} + \frac{g}{k\prime})(1 - e^{-k\prime t}) - \frac{g}{k\prime}t + y_o \tag{26}$$

The t used is the TOF of the bullet, and y_o is the distance from the scope LOS (line of site) to the barrel of the rifle. Also, this equation could be solved to find the initial v_y of the bullet using the TOF for the zero distance (in my example I am using 100 yards). The zero distance is an imaginary point down range the the bullet will pass through y=0. This is also the distance that the scope was calibrated to. Setting $y = 0$ and letting $t = TOF$ for 100 yards. We can do a little algebra and solve for v_y.

$$v_{y_o} = \frac{gt - k\prime y_o}{1 - e^{-k\prime t}} - \frac{g}{k\prime} \tag{27}$$

This has been a long hard journey, but it has been very rewarding in terms of my understanding ballistics. I believe this approach is a little different than others have treated the subject. I have successfully incorporated these equations into a program that produces good output. Also, using these equations have reduced the processing time it takes for my calculator to produce results. I was able to bring the time down from 7 minutes to less than a minute.