STATE SPACE AVERAGING WITH A POCKET CALCULATOR

by Robert R. Boyd
Perkin-Elmer Corp.
Applied Science Division
Pomona, CA

ABSTRACT

The well known capabilities of the state space averaging technique in analyzing switched mode power supplies can be obtained with the HP-28S calculator. This paper shows that the results from [1] can be duplicated with ease with the calculator programs provided herein.

In state space averaging, some algebra is necessary to create the matrices from a given power supply topology. A shortcut method is shown that eliminates much of the algebraic manipulation by utilizing the matrix capabilities of the calculator. The reduction in algebra is directly proportional to the order of the power supply equivalent circuit.

INTRODUCTION

A simplified method of creating the required arrays is shown first using the buck topology as a familiar vehicle to demonstrate the method. The arrays for the boost topolgy are also given followed by a fourth order double-LC topology to demonstrate that the extension to more complex equivalent circuits follows easily.

Coefficients for line-to-output and control-to-output transfer functions, as well as input and output impedance are given for the buck topology. Low resolution plots can be generated by the calculator or the coefficients transferred to any PC plot software.

Finally, writing short calculator subprograms is explained.

CREATING THE ARRAYS

The second order buck topology is shown in figure 1.

The necessary circuit equations are: (interval D, switch on)

$$Ldi/dt + R_1 i + Vs + vo = Vg \qquad (1)$$

$$vo = R_3 i - R_3 Cdv/dt \qquad (2)$$

$$vo = v + R_2 Cdv/dt \qquad (3)$$

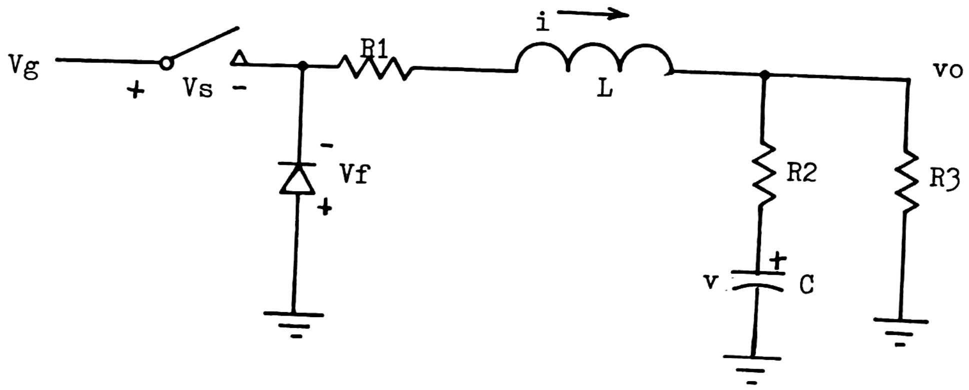

Figure 1. Buck Converter

Substituting (2) into (1) and (3) into (1):

$$L\,di/dt - R_3 C\,dv/dt = Vg - Vs - (R_1 + R_3)i \qquad (4)$$

$$L\,di/dt + R_2 C\,dv/dt = Vg - Vs - R_1 i - v \qquad (5)$$

At this point the algebra would normally continue to get two additional equations of the form di/dt = ... and dv/dt = ... This is relatively easy for two equations, but becomes tedious as the order increases. Hence we terminate the algebraic manipulation when all the Ldi/dt and Cdv/dt terms are on the left hand side, and form matrix equations directly from (4) and (5):

$$\begin{vmatrix} 1 & -R_3 \\ 1 & R_2 \end{vmatrix} \begin{vmatrix} L\,di/dt \\ C\,dv/dt \end{vmatrix} = \begin{vmatrix} -(R_1 + R_3) & 0 \\ -R_1 & -1 \end{vmatrix} \begin{vmatrix} i \\ v \end{vmatrix} + \begin{vmatrix} 1 & -1 & 0 \\ 1 & -1 & 0 \end{vmatrix} \begin{vmatrix} Vg \\ Vs \\ Vf \end{vmatrix}$$

The above equations are of the form:

$$W_1 P\,dx/dt = Q_1 x + S_1 u \quad \text{where} \qquad (6)$$

$$W_1 = \begin{vmatrix} 1 & -R_3 \\ 1 & R_2 \end{vmatrix}, \quad P = \begin{vmatrix} L & 0 \\ 0 & C \end{vmatrix}, \quad dx/dt = \begin{vmatrix} di/dt \\ dv/dt \end{vmatrix},$$

$$Q_1 = \begin{vmatrix} -(R_1 + R_3) & 0 \\ -R_1 & -1 \end{vmatrix}, \quad x = \begin{vmatrix} i \\ v \end{vmatrix}, \quad S_1 = \begin{vmatrix} 1 & -1 & 0 \\ 1 & -1 & 0 \end{vmatrix}, \quad \text{and } u = \begin{vmatrix} Vg \\ Vs \\ Vf \end{vmatrix}.$$

To get the standard form $dx/dt = A_1x + B_1u$ we premultiply (6) by $(W_1P)^{-1}$ and get:

$$dx/dt = (W_1P)^{-1}Q_1x + (W_1P)^{-1}S_1u \quad \text{in which} \qquad (7)$$

$$(W_1P)^{-1}Q_1 = A_1 \text{ and } (W_1P)^{-1}S_1 = B_1. \qquad (8)$$

Thus, once the circuit equations are in the form of (4) and (5), the calculator obtains the required A and B matrices using (8).

During interval $D' = 1 - D$, (switch open) the circuit equations give $W_2 = W_1$, $Q_2 = Q_1$.

Matrix S_2 for interval D' is:

$$S_2 = \begin{vmatrix} 0 & 0 & -1 \\ 0 & 0 & -1 \end{vmatrix}.$$

Hence $A_1 = A_2 = (W_1P)^{-1}Q_1$ and $B_2 = (W_2P)^{-1}S_2$

The output matrix equation (form $y = Ox$) is readily obtained by a method given in [4]:

With capacitor C shorted: $vo' = iR_2//R_3$;

With inductor L open: $vo" = vR_3/(R_2 + R_3)$, and

$$vo = vo' + vo" = \begin{vmatrix} R_2//R_3 & R_3/(R_2 + R_3) \end{vmatrix} \begin{vmatrix} i \\ v \end{vmatrix} = O_1 = O_2.$$

The equations are then averaged as shown in [1]:

$$A = DA_1 + (1-D)A_2,\ B = DB_1 + (1-D)B_2, \text{ etc.}$$

<u>DC SOLUTION</u>

The dc solution is given by setting $WPdx/dt = 0$. Then

$$X = -Q^{-1}SU = -[Q_1D + (1-D)Q_2]^{-1}[S_1D + (1-D)S_2]U.$$

The dc output voltage Vo is obtained from

$$Vo = OX.$$

AC SMALL SIGNAL SOLUTION

The line-to-output and control-to-output transfer functions are obtained by perturbating the variables $x = X + \hat{x}$, $u = U + \hat{u}$, and $d = D + \hat{d}$. This leads to

$$dx/dt = A\hat{x} + B\hat{u} + K\hat{d} + AX + BU, \qquad (9)$$

where $K = (A_1 - A_2)X + (B_1 - B_2)U$

Note: As in [1], the symbols $x = X + \hat{x}$, $u = U + \hat{u}$, etc., are used to designate the dc (uppercase) and small signal ac (lowercase hat) variables.

The last two terms of (9) represent the dc solution which is ignored.

Taking the Laplace transform of (9):

$$\hat{x}(s) = FB\hat{u}(s) + FK\hat{d}(s), \text{ where } F = (sI - A)^{-1} \qquad (10)$$

The output equation $y = Ox$ can sometimes take the form $y = Ox + Eu$. Pertubating this form and taking the Laplace transform results in

$$\hat{y}(s) = O\hat{x}(s) + E\hat{u}(s) + L\hat{d}(s), \qquad (11)$$

where $L = (O_1 - O_2)X$.

Substituting (10) into (11) and dropping the (s) notation gives

$$\hat{y} = OFB\hat{u} + OFK\hat{d} + E\hat{u} + L\hat{d} \qquad (12)$$

The line-to-output transfer function is obtained from (12) by setting $d = 0$, while the control-to-output transfer function is obtained by setting $u = 0$:

$$\hat{y}/\hat{u} = OFB + E \quad \text{(line-to-output)} \qquad (13)$$

$$\hat{y}/\hat{d} = OFK + L \quad \text{(control-to-output)} \qquad (14)$$

Using Leverrier's algorithm [3] the calculator obtains transfer functions for (13) or (14) in the form:

$$\frac{N_{n-1}s^{n-1} + N_{n-2}s^{n-2} + \ldots + N_1 s + N_0}{s^n + D_{n-1}s^{n-1} + \ldots + D_1 s + D_0} \qquad (15)$$

where n is the equivalent circuit order.

BOOST TOPOLOGY

For the boost converter shown in figure 2, the following matrices are obtained for the time periods shown:

Period D (switch on)

$$W_1 = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix}, \quad Q_1 = \begin{vmatrix} -R_1 & 0 \\ 0 & -1/(R_2+R_3) \end{vmatrix}, \quad S_1 = \begin{vmatrix} 1 & -1 & 0 \\ 0 & 0 & 0 \end{vmatrix}.$$

Period 1-D (switch off)

$$W_2 = \begin{vmatrix} 1 & R_2 \\ 0 & 1 \end{vmatrix}, \quad Q_2 = \begin{vmatrix} -R_1 & -1 \\ R_3/(R_2+R_3) & -1/(R_2+R_3) \end{vmatrix},$$

$$S2 = \begin{vmatrix} 1 & 0 & -1 \\ 0 & 0 & 0 \end{vmatrix}.$$

P, x, and u are the same as the buck converter.

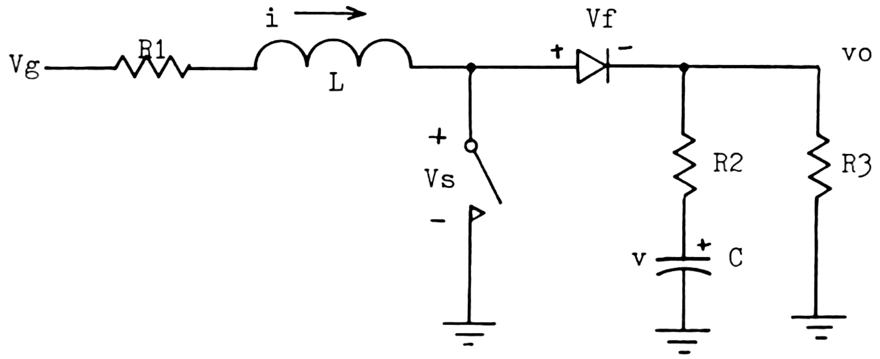

Figure 2. Boost Converter

For the output equation, the matrices O_1 and O_2 for periods D and 1-D respectively are:

$$O_1 = \begin{vmatrix} 0 & R_3/(R_2+R_3) \end{vmatrix}; \quad O_2 = \begin{vmatrix} R_2//R_3 & R_3/(R_2+R_3) \end{vmatrix}.$$

Averaging gives:

$$O = \begin{vmatrix} (1-D)R_2//R_3 & R_3/(R_2+R_3) \end{vmatrix}.$$

DOUBLE LC BUCK TOPOLOGY [2]

For this configuration shown in figure 3, the fourth order arrays are as follows:

Period D (switch on)

$$W_1 = \begin{vmatrix} 1 & R_2 & 0 & 0 \\ 1 & -R_2 & 1 & R_4 \\ 0 & R_5 & 0 & (R_4+R_5) \\ 0 & 0 & 0 & (R_4+R_5) \end{vmatrix}, \quad Q_1 = \begin{vmatrix} -R_1 & -1 & 0 & 0 \\ 0 & 1 & -R_3 & -1 \\ R_5 & 0 & 0 & -1 \\ 0 & 0 & R_5 & -1 \end{vmatrix},$$

$$S_1 = \begin{vmatrix} 1 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{vmatrix}, \quad P = \begin{vmatrix} L_1 & 0 & 0 & 0 \\ 0 & C_1 & 0 & 0 \\ 0 & 0 & L_2 & 0 \\ 0 & 0 & 0 & C_2 \end{vmatrix},$$

$$x = \begin{vmatrix} i_1 & v_1 & i_2 & v_2 \end{vmatrix}^T, \quad u = \begin{vmatrix} Vg & Vs & Vf \end{vmatrix}^T,$$

where ^{T} indicates matrix transpose.

Period 1-D (switch off)

$W_2 = W_1$; $Q_2 = Q_1$.

$$S_2 = \begin{vmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{vmatrix}.$$

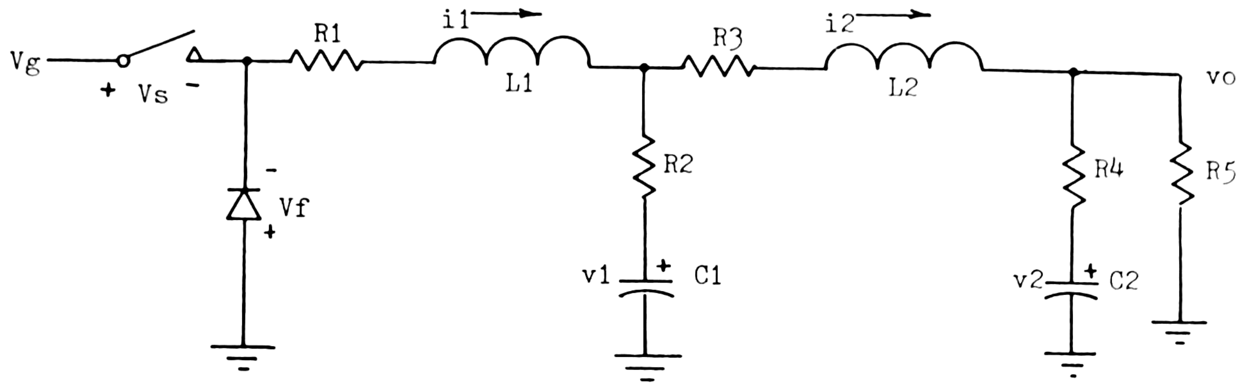

Figure 3. Double LC Buck Converter

For the output equation:

$$O_1 = O_2 = O = \begin{vmatrix} 0 & 0 & R_4//R_5 & R_5/(R_4+R_5) \end{vmatrix}$$

The circuit equations can be reconstructed from the above arrays.

NUMERCAL EXAMPLE

For the buck converter, the following values are assigned:

R1 = 0.3 ohms	Vg = 15 V
R2 = 0.085 ohms	Vs = 0.1 V
R3 = 38.0 ohms	Vf = 0.7 V
L = 0.43 mH	Vm = 8.4 V
C = 47 uF	D = 0.63

The following output was created by the HP-28S programs in Appendix I:

The dc solution is:

$$X = \begin{vmatrix} 0.2383 \\ 9.0565 \end{vmatrix}, \quad Vo = OX = \begin{vmatrix} 0.0848 & 0.9978 \end{vmatrix} \begin{vmatrix} 0.2383 \\ 9.0565 \end{vmatrix} = 9.0565 \text{ V}.$$

For the small signal analysis, the line-to-output transfer function (13) is:

$$y(s)/u(s) = vo(s)/vg(s) = \frac{N_1 s + N_0}{D_2 s^2 + D_1 s + D_0.} \qquad (16)$$

The Bode plot for this transfer function is shown in figure 4.

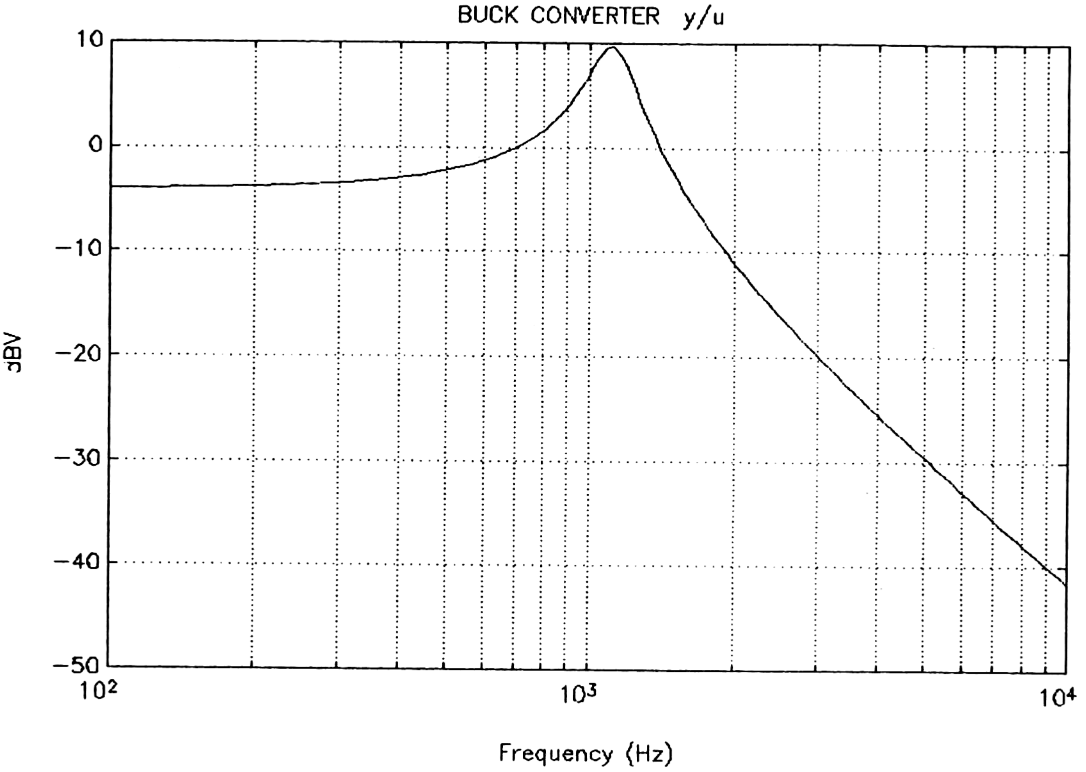

Figure 4.

(Note: The plots are from a PC plot program using coefficients computed by the calculator.)

Table I was created to show the various transfer function parameters including that shown in figure 4:

TABLE I

	BUCK y/u	BOOST y/u	BUCK y/d	BOOST y/d
Vo	9.057	37.507	9.057	37.507
N_1	1.243E2	7.298E1	3.663E2	-6.395E3
N_0	3.110E7	1.827E7	9.169E7	7.818E7
D_2	1	1	1	1
D_1	1.454E3	1.329E3	1.454E3	1.329E3
D_0	4.976E7	7.174E6	4.967E7	7.174E6
Zero	-3.984E4	-3.984E4	-3.984E4	+1.946E3
Pole	1.123E3	4.263E2	1.123E3	4.263E2

INPUT ADMITTANCE [1]

Equation (13) is a transfer (function) matrix of dimensions number of outputs by number of inputs. Hence (16) is actually $G_{11}(s)$ in the transfer matrix (dropping (s) notation):

$$G = \begin{vmatrix} G_{11} & G_{12} & G_{13} \end{vmatrix} = \begin{vmatrix} vo/vg & vo/vs & vo/vf \end{vmatrix}$$

For the buck converter, input admittance is obtained by including the input current i_{IN} as part of the output equation y = Ox:

$$\begin{vmatrix} i_{IN} \\ vo \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ R_2//R_3 & R_3/(R_2+R_3) \end{vmatrix} \begin{vmatrix} i \\ v \end{vmatrix} \qquad (y = O_1x)$$

$$\begin{vmatrix} i_{IN} \\ vo \end{vmatrix} = \begin{vmatrix} 0 & 0 \\ R_2//R_3 & R_3/(R_2+R_3) \end{vmatrix} \begin{vmatrix} i \\ v \end{vmatrix} \qquad (y = O_2x)$$

Transfer matrix G is then:

$$\begin{vmatrix} i_{IN}/vg & i_{IN}/vs & i_{IN}/vf \\ vo/vg & vo/vs & vo/vf \end{vmatrix}$$

Element G_{11} is thus the input admittance of the buck converter shown in figure 5. For this plot N_1 = 9.23E2, N_0 = 5.157E5, D_2 = 1, D_1 = 1.454E3, and D_0 = 4.976E7.

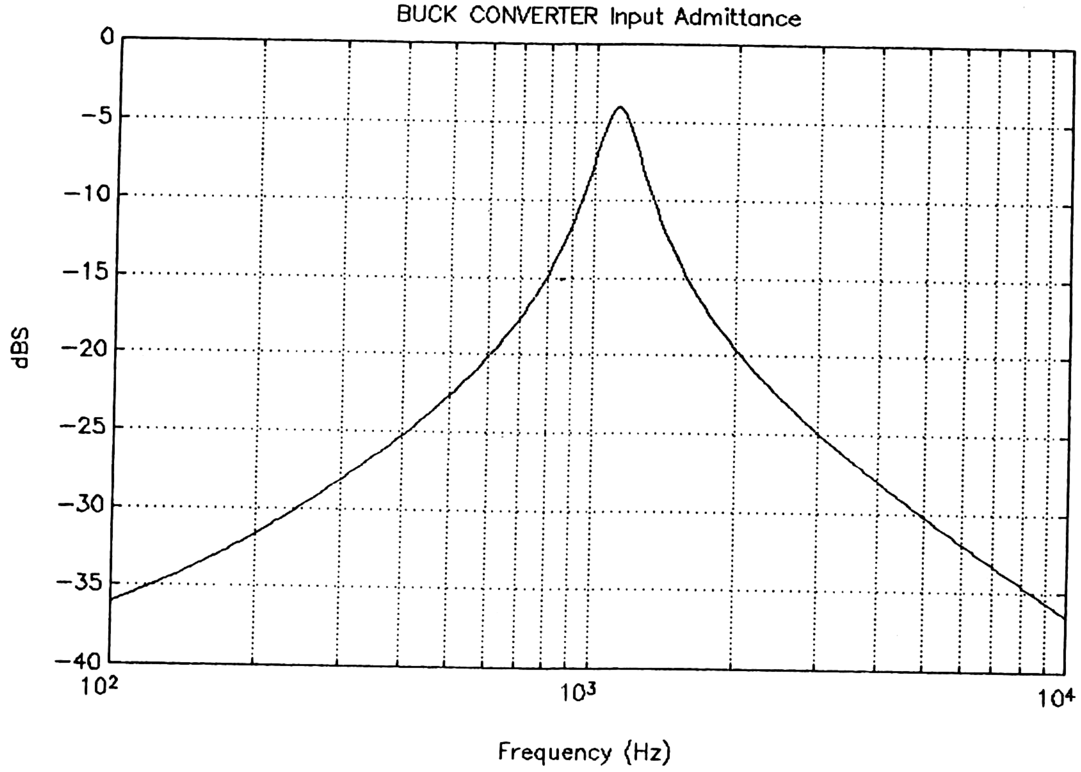

Figure 5.

OUTPUT IMPEDANCE [1]

To get output impedance Zo insert an independent ac current generator i_G (I_G = 0) in the buck regulator output node. Matrices S_1 and S_2 become:

$$S_1 = \begin{vmatrix} 1 & -1 & 0 & -R_3 \\ 1 & -1 & 0 & 0 \end{vmatrix}, \quad S_2 = \begin{vmatrix} 0 & 0 & -1 & -R_3 \\ 0 & 0 & -1 & 0 \end{vmatrix}$$

and $u = \begin{vmatrix} Vg & Vs & Vf & i_G \end{vmatrix}^T$. Since there is now an input coupled directly to the output, the output equation is $y = 0x + Eu$ where

$$E = \begin{vmatrix} 0 & 0 & 0 & R_2//R_3 \end{vmatrix}.$$

Transfer matrix G is:

$$\begin{vmatrix} vo/vg & vo/vs & vo/vf & vo/i_G \end{vmatrix}$$

and $Zo = G_{14} = vo/i_G$. See figure 6 in which N_1 = 2.118E3, N_0 = 1.478E7. The D coefficients are unchanged from figure 5.

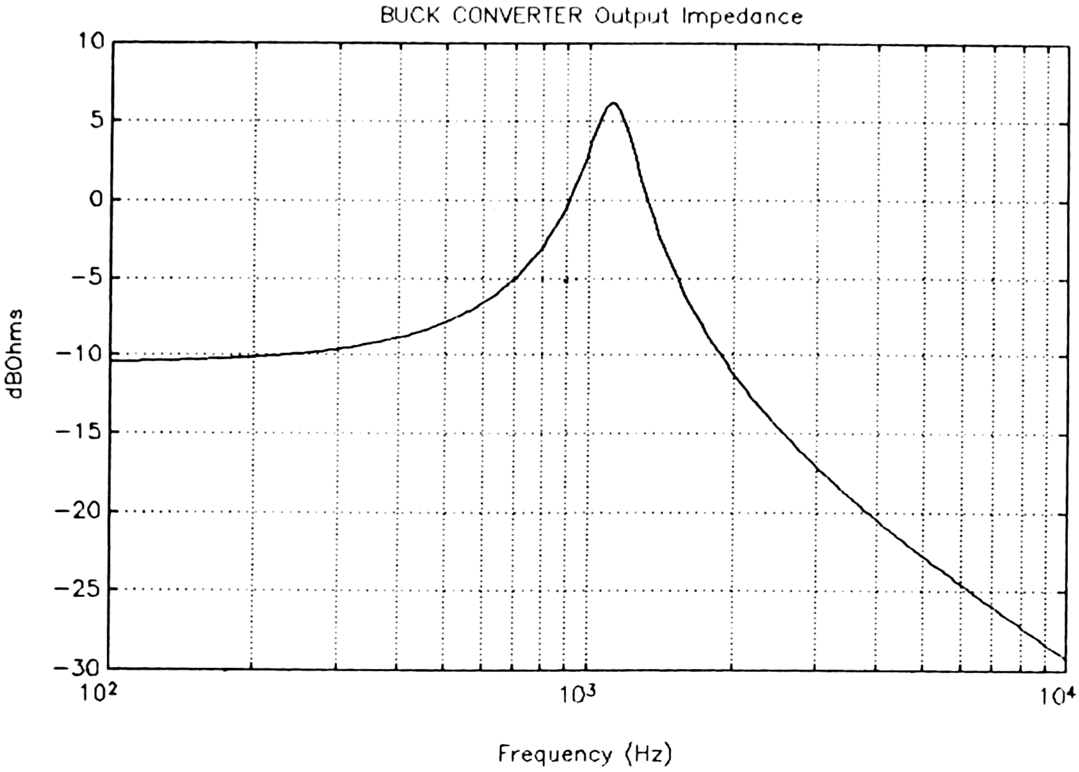

Figure 6.

CREATING CALCULATOR SUBPROGRAMS

The complete listing of the calculator main programs is given in the Appendix. Here we show how to write the subprograms unique to each equivalent circuit.

The subprograms are the W, P, Q, S, & O array elements written in row order. For example, to create W1 for the buck converter, a subprogram labled "SUW1" (for Set Up W1) is

<< 1 R3 NEG 1 R2 >>.

Matrix Q1 for the double LC topology is similarly written as "SUQ1":

<< R1 NEG 1 NEG 0 0 0 1 R3 NEG 1 NEG R5 0 0 1 NEG 0 0 R5 1 NEG >>

Subprogram "NMK" provides the array dimensions of N (order), M (no. of inputs), and K (no. of outputs), and for the input admittance example is:

<< 2 3 2 'K' STO 'M' STO 'N' STO >>

REFERENCES

[1] F. Barzegar, R.D. Middlebrook and S. Cuk, "Using Small Computers to Model and Measure Magnitude and Phase of Regulator Transfer Functions and Loop Gain", Advances in Switched-Mode Power Conversion, Vols I & II, TESLACo, 1983.

[2] R.B. Ridley, "Secondary LC Filter Analysis and Design Techniques for Current-Mode-Controlled Converters", IEEE Transactions on Power Electronics, Vol 3, No. 4, 4 Oct 1983

[3] D. M. Wiberg, "State Space and Linear Systems", Schaums Outline Series, McGraw-Hill, 1971

[4] P.M. DeRusso, R.J. Roy, & C.M. Close, "State Variables for Engineers", Wiley, 1965

APPENDIX

Calculator Program Listings

```
BEGIN
« LVL3 CLLCD
  IF 1 FC?
  THEN "y/u"
  ELSE "y/d"
  END 1 DISP DCSLN
FMAB SMA PIK PURG UP
»

DCSLN
« NMK SUQ1 N N TLA
SUQ2 N N TLA SUS1 N
M TLA SUS2 N M TLA
Vg Vs Vf M 1 TLA 'U'
STO → q1 q2 s1 s2
  « s1 s2 MAVG q1 q2
MAVG / U * NEG 'X'
STO SUO1 K N TLA
SUO2 K N TLA 'O2'
STO 'O1' STO O1 O2
MAVG 'O' STO O X *
ARRY→ DROP 3 FIX
→STR "Vo = " SWAP +
4 DISP CLEAR q1 q2
s1 s2
  »
»

FMAB
« → g1 q2 s1 s2
  « SUW1 N N TLA
SUW2 N N TLA → w1 w2
    « q1 w1 / 'A1'
STO q2 w2 / 'A2' STO
A1 A2 MAVG 'A' STO
s1 w1 / 'B1' STO s2
w2 / 'B2' STO B1 B2
MAVG 'B' STO SUP N N
TLA → p
      « 'A' p STO/
'A1' p STO/ 'A2' p
STO/ 'B' p STO/ 'B1'
p STO/ 'B2' p STO/
      »
    »
  »
»

SMA
« N K *
  IF 1 FS?
  THEN 1
  ELSE M
  END 2 →LIST 0 CON
'G' STO N 1 →LIST 0
CON 'T1' STO N IDN
'ID' STO 1 N
  FOR q
    IF 1 g ==
    THEN ID 'F' STO
q OFB
    END A F * 'AFn'
STO 0 'T' STO q GTT
    IF q N ≠
    THEN 'T1' q GET
ID * AFn + 'F' STO q
1 + OFB
    END
  NEXT
»

OFB
« 1 - K * → j1
  «
    IF 1 FC?
    THEN 'O*F*B'
    ELSE '(O*F*((A1-
A2)*X+(B1-B2)*U)+(O1
-O2)*X)/Vm'
    END EVAL ARRY→
DROP K 1
    FOR i M 1
      FOR j 'G' i j1
+ j 2 →LIST ROT PUT
-1
      STEP -1
    STEP
  »
»

GTT
« → q
  « 1 N
    FOR i 'AFn' i
DUP 2 →LIST GET 'T'
STO+
    NEXT 'T1' q T q
/ NEG PUT
  »
»

MAVG
« 1 D - * SWAP D * +
»

CTS
« SWAP RND STD →STR
+
»

TLA
« 2 →LIST →ARRY
»

PIK
« 'G' 'G'
  IF 2 K ==
  THEN { 3 1 }
  ELSE { 2 1 }
  END { 1 1 } ROT
SWAP GET 'N1' STO
GET 'N0' STO T1
ARRY→ DROP 'D0' STO
'D1' STO
»

NMK
« 2 3 2 'K' STO 'M'
STO 'N' STO
»

TFFR
« LVL3 { STO BF PD
ND } MENU HALT BF ND
+ 'FL' STO CLEAR BF
FL
  FOR f f TFCN R→P
C→R SWAP LOG 20 * f
2 FIX →STR " " +
SWAP 3 FIX RND →STR
+ " " + SWAP 0 FIX
RND →STR + PD INV
  STEP DEPTH →LIST
'VOUT' STO VOUT 3
FIX UP 23 MENU
»

TFCN
« ALOG 2 π * * 0
SWAP R→C → s
  « '(N3*s^3+N2*s^2+
N1*s+N0)/(D4*s^4+D3*
s^3+D2*s^2+D1*s+D0)'
EVAL
  »
»

BPLT
« LVL3 BF SWAP R→C
PMIN FL SWAP R→C
PMAX BF 0 R→C AXES
CLLCD DRAX 'VOUT' 1
1 →LIST BF FL
  START GETI STR→
    IF 2 FS?
    THEN SWAP
    END DROP R→C
PIXEL PD INV
  STEP DROP2 UP
»
```