1 Buffon's Needle

Probability theory is the mathematics of the 20^{th} century. Its history goes back to the 16^{th} century, but not until the previous century did physicists and engineers fully realize that nature and the real world can be described exhaustively only by the laws governing their randomness. What physicists had considered exact until relatively recently, turned out to be merely the mean value of a much more impressive structure; and mean values can be very misleading. ("Put one foot in an ice bucket, and the other in boiling water; then on the average you will be comfortable.") Strange to relate, even as brilliant a physicist as Albert Einstein regarded the probalistic laws of quantum mechanics as testimony to our ignorance rather than as a valid description of the laws of nature.

The beginnings of probability theory go back to the *Liber de ludo aleaæ* (The book of games of chance), written about 1526 by Gerolamo Cardono (1501-1576), though not published until 1663. Cardano, of cubic equation fame, was not only a mathematician, engineer, and physician, but also a passionate gambler. Until the advent of the kinetic theory of gases in the 19^{th} century, probability theory was rarely applied to anything else but gambling. The main contributors to its development were Jacques Bernoulli I (1654-1705, author of *Ars conjectandi*, Blaise Pascal (1623-1662, discoverer of the Pascal Triangle), Abraham De Moivre (1667-1754), Leonhard Euler (1707-1783), Pierre Simon Laplace (1749-1827), Carl Friedrich Gauss (1777-1855), and Sim''eon Denis Poisson (1781-1840), followed by a large number of mathematicians in the 19^{th} and 20^{th} centuries.

The number π appears in probability theory very frequently, as it does in all branches of higher mathematics; but nowhere is its appearance more fascinating than in a problem posed and solved by George Louis Leclerc, Comte du Buffon (1707-1788). Buffon (as everybody calls him) was an able mathematician and general scintist, who shocked the world by estimating the age of the earth to be about 75,000 years, although every educated person in the 18^{th} century knew that it was no older than about 6,000 years. Among his exploits is a test of one of Archimedes' supposed engines of war used in the defense of Syracuse. As told by Plutarch, the story includes a plausible description of the action of Archimedes' cranes and missile throwers, but by the Middle Ages, it had grown into a much exaggerated legend, and the *Book of Histories* by the Byzantine author John Tzetzes (ca. 1120-1183) repeats the story with many embellishments, such as the statement that Archimedes had burned the Roman ships to ashes at a distance of a bow shot by focusing the sun's beams onto the Roman fleet. The story (which is not contained n Plutarch's description) has persisted in many books down to our own day. Buffon, a man of considerable means and spare time, decided to test the feasibility of such a machine. Using 168 flat mirrors six by eight inches in an adjustable framework, he was able to ignite wooden planks at a distance of 150 feet, and he satisfied himself that Archimedes' alleged exploit was feasible. He did not, however, satisfy posterity, since the Syracusans would hardly have had the same leisure to focus 168 beams, nor would the Roman ships floating on the sea have held as still as Buffon's beams on the ground.

But back to Buffon's problem involving π. The problem which he posed (and solved) in 1777 was the following: Let a needle of length L be thrown at random onto a horizontal plane ruled with parallel straight lines spaced by a distance d (greater than L) from each other. What is the probability that the needle will intersect one of these lines?

We assume that "at random" means that any position (of the center) and any orientation of the needle are equally probable and that these two random variables are independent. Let the distance of the center of the needle from the nearest line be x, and let its orientation be given by ϕ (figure 1). Since x is measured from the *nearest* line, we need only consider a single line, because the others involve only repetition of the same solution.

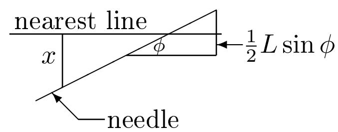


Figure 1: Buffon's needle

It is obvious from the figure that the needle will intersect a line if and only if

$$x < \frac{1}{2}L\sin\phi \tag{1}$$

The problem is therefore equivalent to finding the probability

$$P(x < \frac{1}{2}L\sin\phi)$$

To find this probability, use the plane of rectangular coordinates ϕ, x, and consideer the interior of the rectangle OA (figure2) whose points satisify the inequalities

$$0 < x < \frac{d}{2} \tag{2}$$

$$0 < \phi < \pi$$

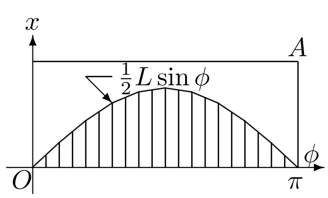


Figure 2: Buffon's problem

These are the intervals of possible values of x and ϕ, and therefore any point inside rectangle OA corresponds to one and only one possible combination of position (x) and orientation (ϕ) of the needle. Since all such combinations are equiprobable, and the area of the rectangle represents the sum total of all possibilities that can arise (because, not quite beyond reproach, we regard this area as made up of all points inside it). However, not all of these possibilities will result in an intersection of the needle with a line; such an intersection, as we have found, will take place only under condition (1), that is, for positions and orientations corresponding to points lying below the curve $x = \frac{1}{2}L\sin\phi$ in Figure 2, so that the sum total of possibilities resulting in the intersection by the needle is given by the area under this curve. If, then, probability is the ratio of the number of favorable, to the number of possible, events under given conditions, the probability of intersection is given by the ratio of the shaded part to the entire rectangle OA in Figure 2, that is, the required probability (2) is

$$P = \frac{1}{2}L\int_0^\pi \sin\phi \, d\phi : \frac{\pi d}{2} = \frac{2L}{\pi d} \tag{3}$$

This is the result Buffon derived. He also attempted an experimental verification of his result by throwing a needle many times onto ruled paper and observing the fraction of intersections out of all throws. Whether he modified his result for an evaluation of π we do not know, but the problem and its solution were largely forgotten for the next 35 years, until one of the great mathematicians with whom France has been blessed, called attention to it and gave it a new twist.

Pierre Simon Laplace was one of the "three great L's" among French mathematicians of the time. The other two, Joseph Louis Lagrange (1736-1813) and Adrien Marie Legendre (1752-1833), were his contemporaries, and all three survived the French Revolution as members of the Committee of Weights and Measures, which discarded the cubits, feet, pounds, and miles of the old regime and worked out the metric system as we use it today. It was, incidentally, another mathematician, Lazare Carnot (1753-1823) who saved the young French republic in its hour of greatest need. Scared out of their wits by the cry for liberty, equality, and fraternity, Europe's kings, princes, princelings, dukes, and whatnots turned on the Revolution. Threatened by internal confusion and the invading armies deep inside France, the Revolution seemed about to be crushed; but Carnot, member of the Committee for Public Safety in charge of military affairs, took command and sent the invaders packing on all fronts, becoming *organisateur de la victorire*, the hero of the French Revolution. But like so many other sincere revolutionaries after him, Carnot soon observed that a revolution only replaces one tyranny by another, and refusing to go along with its excesses, was driven into exile as a "royalist." Significantly, his chair of geometry at the *Institut National* was unanimously voted to a general; a general by the name of Napoleon Bonaparte, another one in a long line of power-hungry careerists who was to preach liberty and practice oppression.

Laplace is known, above all, for authoring two masterpieces, *Méchanique céleste* (five volumes, 1799-1825) and *Théorie analytique des probabilites* (1812). The former was the greatest work on celestial mechanics since Newton's *Principia*, including many new mathematical techniques, such as the theory of potential. Asked by Napoleon why in the entire work on celestial mechanics he had not once mentioned God, Laplace replied, *Sire, je n'avais pas besoin de cette hypothèse*—Sire, I had no need of that hypothesis. Napoleon, incidentally, appointed Laplace Minister of Interior, but after six weeks dismissed him again, commenting that he "carried the spirit of the infinitely small into the management of affairs." The *Théorie analytique* is the foundation of modern probability theory. Among many new mathematical techniques it contains the integral transform that is today the daily bread of every systems engineer and analyst of electrical circuits.

It also contains a discussion of Buffon's problem, which Laplace saw in a new light. From the first and last expressions in (3) we have

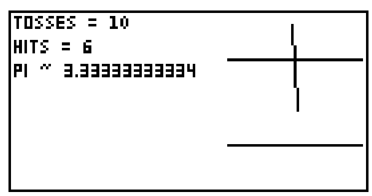
$$\pi = \frac{2L}{dP} \tag{4}$$

and this is an entirely new method of evaluating π: The length of the needle L and the spacing between the lines d are known (usually one makes $L = d$), and the probability of intersection P can be measured by throwing a needle onto ruled paper a very large number of times, recording the fraction of throws resulting in an intersection of the needle with a line.

This method, which Laplace generalized for paper with two sets of mutually perpendicular lines, has been used by several people as a playful diversion to calculate the first decimal places of π by thousands of throws. One of them was a certain Captain Fox, who indulged in this sport while recovering from wounds incurred in the American Civil War.

2 The Buffon Aplet

Transfer the aplet to your `HP 39g` using whatever method you normally use for transferring aplets to your calculator.



Upon starting, the aplet simply prompts you to press ENTER for each toss of the needle. Each toss places the center of the needle between the two lines, oriented at a random angle. The number of tosses so far is displayed and, if the number of "hits" or line crossings is not zero, it and the current estimate of π are also displayed.

Don't be surprised if two different runs produce two different results. The aplet does not reset the random number generator, so each run starts with the current machine state.