1 Buffon's Needle

Probability theory is the mathematics of the 20^{th} century. Its history goes back to the 16^{th} century, but not until the previous century did physicists and engineers fully realize that nature and the real world can be described exhaustively only by the laws governing their randomness. What physicists had considered exact until relatively recently, turned out to be merely the mean value of a much more impressive structure; and mean values can be very misleading. ("Put one foot in an ice bucket, and the other in boiling water; then on the average you will be comfortable.") Strange to relate, even as brilliant a physicist as Albert Einstein regarded the probalistic laws of quantum mechanics as testimony to our ignorance rather than as a valid description of th laws of nature.

The beginnings of probability theory go back to the *Liber de ludo aleaæ* (The book of games of chance), written about 1526 by Gerolamo Cardono (1501-1576), though not published until 1663. Cardano, of cubic equation fame, was not only a mathematician, engineer, and physician, but also a passionate gambler. Until the advent of the kinetic theory of gases in the 19^{th} century, probability theory was rarely applied to anything else but gambling. The main contributors to its development were Jacques Bernoulli I (1654-1705, author of *Ars conjectandi*, Blaise Pascal (1623-1662, discoverer of the Pascal Triangle), Abraham De Moivre (1667-1754), Leonhard Euler (1707-1783), Pierre Simon Laplace (1749-1827), Carl Friedrich Gauss (1777-1855), and Sim''eon Denis Poisson (1781-1840), followed by a large number of mathematicians in the 19^{th} and 20^{th} centuries.

The number π appears in probability theory very frequently, as it does in all branches of higher mathematics; but nowhere is its appearance more fascinating than in a problem posed and solved by George Louis Leclerc, Comte du Buffon (1707-1788). Buffon (as everybody calls him) was an able mathematician and general scintist, who shocked the world by estimating the age of the earth to be about 75,000 years, although every educated person in the 18^{th} century knew that it was no older than about 6,000 years. Among his exploits is a test of one of Archimedes' supposed engines of war used in the defense of Syracuse. As told by Plutarch, the story includes a plausible description of the action of Archimedes' cranes and missile throwers, but by the Middle Ages, it had grown into a much exaggerated legend, and the *Book of Histories* by the Byzantine author John Tzetzes (ca. 1120-1183) repeats the story with many embellishments, such as the statement that Archimedes had burned the Roman ships to ashes at a distance of a bow shot by focusing the sun's beams onto the Roman fleet. The story (which is not contained n Plutarch's description) has persisted in many books down to our own day. Buffon, a man of considerable means and spare time, decided to test the feasibility of such a machine. Using 168 flat mirrors six by eight inches in an adjustable framework, he was able to ignite wooden planks at a distance of 150 feet, and he satisfied himself that Archimedes' alleged exploit was feasible. He did not, however, satisfy posterity, since the Syracusans would hardly have had the same leisure to focus 168 beams, nor would the Roman ships floating on the sea have held as still as Buffon's beams on the ground.

But back to Buffon's problem involving π. The problem which he posed (and solved) in 1777 was the following: Let a needle of length L be thrown at random onto a horizontal plane ruled with parallel straight lines spaced by a distance d (greater than L) from each other. What is the probability that the needle will intersect one of these lines?

We assume that "at random" means that any position (of the center) and any orientation of the needle are equally probable and that these two random variables are independent. Let the distance of the center of the needle from the nearest line be x, and let its orientation be given by ϕ (figure 1). Since x is measured from the *nearest* line, we need only consider a single line, because the others involve only repetition of the same solution.

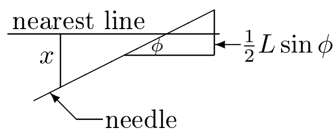

Figure 1: Buffon's needle

It is obvious from the figure that the needle will intersect a line if and only if

$$x < \frac{1}{2}L\sin\phi \tag{1}$$

The problem is therefore equivalent to finding the probability

$$P(x < \frac{1}{2}L\sin\phi)$$

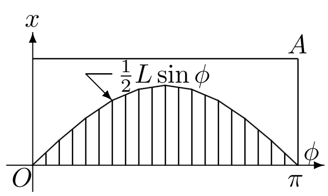

Figure 2: Buffon's problem

To find this probability, use the plane of rectangular coordinates ϕ, x, and consideer the interior of the rectangle OA (figure2) whose points satisify the inequalities

$$0 < x < \frac{d}{2} \tag{2}$$

$$0 < \phi < \pi$$

These are the intervals of possible values of x and ϕ, and therefore any point inside rectangle OA corresponds to one and only one possible combination of position (x) and orientation (ϕ) of the needle. Since all such combinations are equprobable, and the area of the rectangle represents the sum total of all possibilities that can arise (because, not quite beyond reproach, we regard this area as made up of all points inside it). However, not all of these possibilities will result in an intersection of the needle with a line; such an intersection, as we have found, will take place only under condition (1), that is, for positions and orientations corresponding to points lying below the curve $x = \frac{1}{2}L\sin\phi$ in Figure 2, so that the sum total of possibilities resulting in the intersection by the needle is given by the area under this curve. If, then, probability is the ratio of the number of favorable, to the number of possible, events under given conditions, the probability of intersection is given by the ratio of the shaded part to the entire rectangle OA in Figure 2, that is, the required probability (2) is

$$P = \frac{1}{2}L\int_0^\pi \sin\phi \; d\phi : \frac{\pi d}{2} = \frac{2L}{\pi d} \tag{3}$$

This is the result Buffon derived. He also attempted an experimental verification of his result by throwing a needle many times onto ruled paper and observing the fraction of intersections out of all throws. Whether he modified his result for an evaluation of π we do not know, but the problem and its solution were largely forgotten for the next 35 years, until one of the great mathematicians with whom France has been blessed, called attention to it and gave it a new twist.

Pierre Simon Laplace was one of the "three great L's" among French mathematicians of the time. The other two, Joseph Louis Lagrange (1736-1813) and Adrien Marie Legendre (1752-1833), were his contemporaries, and all three survived the French Revolution as members of the Committee of Weights and Measures, which discarded the cubits, feet, pounds, and miles of the old regime and worked out the metric system as we use it today. It was, incidentally, another mathematician, Lazare Carnot (1753-1823) who saved the young French republic in its hour of greatest need. Scared out of their wits by the cry for liberty, equality, and fraternity, Europe's kings, princes, princelings, dukes, and whatnots turned on the Revolution. Threatened by internal confusion and the invading armies deep inside France, the Revolution seemed about to be crushed; but Carnot, member of the Committee for Public Safety in charge of military affairs, took command and sent the invaders packing on all fronts, becoming *organisateur de la victorire*, the hero of the French Revolution. But like so many other sincere revolutionaries after him, Carnot soon observed that a revolution only replaces one tyranny by another, and refusing to go along with its excesses, was driven into exile as a "royalist." Significantly, his chair of geometry at the *Institut National* was unanimously voted to a general; a general by the name of Napoleon Bonaparte, another one in a long line of power-hungry careerists who was to preach liberty and practice oppression.

Laplace is known, above all, for authoring two masterpieces, *Méchanique céleste* (five volumes, 1799-1825) and *Théorie analytique des probabilites* (1812). The former was the greatest work on celestial mechanics since Newton's *Principia*, including many new mathematical techniques, such as the theory of potential. Asked by Napoleon why in the entire work on celestial mechanics he had not once mentioned God, Laplace replied, *Sire, je n'avais pas besoin de cette hypothèse*—Sire, I had no need of that hypothesis. Napoleon, incidentally, appointed Laplace Minister of Interior, but after six weeks dismissed him again, commenting that he "carried the spirit of the infinitely small into the management of affairs." The *Théorie analytique* is the foundation of modern probability theory. Among many new mathematical techniques it contains the integral transform that is today the daily bread of every systems engineer and analyst of electrical circuits.

It also contains a discussion of Buffon's problem, which Laplace saw in a new light. From the first and last expressions in (3) we have

$$\pi = \frac{2L}{dP} \tag{4}$$

and this is an entirely new method of evaluating π: The length of the needle L and the spacing between the lines d are known (usually one makes $L = d$), and the probability of intersection P can be measured by throwing a needle onto ruled paper a very large number of times, recording the fraction of throws resulting in an intersection of the needle with a line.

This method, which Laplace generalized for paper with two sets of mutually perpendicular lines, has been used by several people as a playful diversion to calculate the first decimal places of π by thousands of throws. One of them was a certain Captain Fox, who indulged in this sport while recovering from wounds incurred in the American Civil War.

It is not difficult to calculate the probability of obtaining π correct to k decimal places in N throws. The results of such a calculation show that this method is very inefficient as far as the numerical computation of π is concerned. Nevertheless, Laplace had discovered a powerful method of computation that did not come into its own until the advent of the electronic computer. The method that Laplace proposed consists in finding a numerical value by realizing a random event many times and observing its outcome experimentally. This is today known as a Monte Carlo method (Monte Carlo is the European Las Vegas), and it is used in a wide field of applications ranging from economics to nuclear physics.

But if the method is not very efficient for calculating π, it is very powerful in other applications. Suppose, for example, that we wish to calculate the mean value of a complicated function of a random variable. This is found by an integration involving the probability density function of the random variable. But sometimes the resulting integral is so complicated that it takes a long time to write the program and that it involves a costly amount of processing time. In that case we do not program the computer for the complicated evaluation of the integral, but we make it simulate the arithmetic mean of, say, one hundred thousand trials. The result is the required mean value.

Or suppose we wish to find a complicated multiple integral. A Monte Carlo method of finding it (instead of writing a cumbersome program) is to let the computer "shoot" n-tuplets of random numbers. These represent a coordinate in (n-dimensional) space and the coordinate either lies in the volume determined by the integral ("hit") or it does not ("miss"). Then we let the computer shoot at the target, say, half a million times. The number of hits is then proportional to the n-tuple integral.

The man who taught us to program electronic computers in this way was Pierre Simon Laplace. His computer was neither electronic nor digital. It was an analog computer consisting of one needle and one piece of ruled paper.

2 Simulation Programs

Buffon's Needle can be implemented easily on all but the smallest HP programmable calculators (the `HP-16` and the financial models will need an implementation of the sine function).

2.1 RPN

Some of the earlier machines did not have a built-in RNG (random number generator). One can be programmed, of course, but that requires additional memory. In addition, the HP-25A/C and HP-33E/C don't have enough memory registers unless one packs multiple results into each register (which requires additional code).

The program given here is written for the HP-15C. It takes advantage of the built-in RNG and the cability to `DSE` any register. Every other model lacks one or both of these capabilities, though neither one by itself takes much programming effort to overcome.

001 <u>LBL B</u>	012 <u>LBL 3</u>	023 ISG (i)	034 ×
002 STO .2	013 RAN#	024 <u>LBL .0</u>	035 +
003 ST+ .3	014 RAN#	025 DSE .2	036 DSE I
004 <u>LBL 2</u>	015 π	026 GTO 2	037 GTO 1
005 RCL .2	016 ×	027 1	038 RCL .3
006 PSE	017 SIN	028 0	039 $x \rightleftharpoons y$
007 CLx	018 $x > y$?	029 STO I	040 ÷
008 STO I	019 ISG I	030 CLx	041 2
009 1	020 <u>LBL .0</u>	031 <u>LBL 1</u>	042 0
010 0	021 DSE .1	032 RCL I	043 ×
011 STO .1	022 GTO 3	033 RCL (i)	044 RTN
Step 018 is actually `TEST 7`			

The memory registers should be cleared before running the program the first time. (Actually, only registers 0 through .0 and .3 need to be clear.) The program requires `RAD` mode; if you prefer, you may insert a `RAD` instruction before the `LBL 2`. Enter the desired number of trials (of ten throws each) into `X` and press `GSB B`. It displays a countdown timer showing the number of trials yet to be done. When finished, it displays the current estimate of π. The number of trials that resulted in 0 intersections is stored in `R0`, the number of trials that resulted in 1 intersection is stored in `R1`, and so on up to 10 intersections in `R.0`. The number of trials is stored in `R.3`.

If the program is repeated, the new results are added to the previous ones. This allows a big run to be broken up into smaller runs.

This is not a fast program, requiring about 18 seconds for each trial of ten tosses on the author's emulated `HP-15C`, or not quite 200 trials per hour. A HP-41CX version is a little faster, requiring about 15 seconds for each trial, or about 240 trials per hour.

2.2 BASIC

For the HP-71B the following BASIC program simulates the experiment. It prompts for the number of ten-throw trials, then prints the occurences of each outcome and an estimate for π.

```
10 DESTROY ALL @ OPTION BASE 0 @ RADIANS @ DIM R(11) @ P=0
20 INPUT N @ FOR I=1 TO N
30 H=0 @ FOR J=1 TO 10 @ IF RND()<SIN(RND()*PI) THEN H=H+1
40 NEXT J @ R(H)=R(H)+1 @ P=P+H
50 NEXT I
60 FOR I=0 TO 10 @ PRINT I;R(I) @ NEXT I
70 PRINT 20*N/P
```

Having neither a physical HP-71B nor an emulator that runs at actual speed, the author was unable to get actual running times for this program.

2.3 RPL

Using RPL for the HP 48 series, the program practically writes itself from the problem description, doing everything on the stack and using a vector instead of numbered registers.

```
«
 0 11 FOR j
  j
 NEXT
 →LIST { 11 } 0 CON
 1 4 PICK START
  1
  1 10 START
   RAND RAND π * SIN < +
  NEXT
  DUP2 GET 1 + PUT
 NEXT
 OBJ→ DROP 11 →LIST SWAP OVER * ΣLIST ROT 20 * SWAP /
»
```

Like the RPN version, this program takes the desired number of ten-throw trials on the stack. Also like the RPN version, it expects to be in RAD mode. It also expects symbolic mode to be off. It finishes with an estimate for π on level 1 and a list holding the results on level 2. Unlike the RPN version, a second run of this program does not add to a cumulative total.

Due to advances in calculator technology, this program is much faster, doing nearly three trials per second on the author's physical HP 48GX, or just over 10,000 trials per hour.

3 Expected Outcomes

Below are the results from actual runs on an emulated HP-15C (200 trials), an emulated HP-41CX (300 trials), an emulated HP-71B (10,000) trials, and a physical HP 48GX (10,000 trials).

Machine	Intersections										
	0	1	2	3	4	5	6	7	8	9	10
15C	0	1	2	3	18	28	55	42	34	14	3
41CX	0	0	1	5	23	55	79	77	48	22	1
71B	0	7	55	239	813	1658	2469	2357	1625	660	117
48GX	0	5	50	287	778	1693	2399	2444	1635	601	108

Their estimates for π are given below.

Machine	estimated π
15C	3.129890454
41CX	3.138075314
71B	3.134845374
48GX	3.143962021

Obviously, this is not a great way to calculate π. But that really wasn't the point of the exercise.

Probability theory tells us how often we may expect each outcome. Buffon's Needle is a classic case of a *Bernoulli experiment*, in which there are just two outcomes of interest—the needle either intersects a line or it doesn't. The *binomial formula* tells us

$$P(S_n = k) = P(k \text{ successes in } n \text{ trials}) = \binom{n}{k} p^k q^{n-k}$$

where $q = 1 - p$.

This RPL program creates a list containing the eleven probabilities for the problem at hand, $n = 10, p = 2/\pi$.

```
«
 2 π / 1 OVER - → p q
 <<
  { }
  0 10 FOR j
   10 j COMB p j ^ * q 10 j - ^ * +
  NEXT
 >>
»
```

The output of this program, rounded to five decimal places, is given in the table below.

k	0	1	2	3	4	5
$P(k)$	0.00004	0.00070	0.00554	0.02590	0.07942	0.16696
k	6	7	8	9	10	
$P(k)$	0.24375	0.24402	0.16032	0.06241	0.01093	

The table below shows the observed and expected values for the four sets of experiments.

Machine	0	1	2	3	4	5	6	7	8	9	10
15C	0	1	2	3	18	28	55	42	34	14	3
	0	0	1	5	16	33	49	49	32	12	2
41CX	0	0	1	5	23	55	79	77	48	22	1
	0	0	2	8	24	50	73	73	48	19	3
71B	0	7	55	239	813	1658	2469	2357	1625	660	117
	0	7	55	259	794	1670	2438	2440	1603	624	109
48GX	0	5	50	287	778	1693	2399	2444	1635	601	108
	0	7	55	259	794	1670	2438	2440	1603	624	109

The observed values seem to match "fairly" well with the expected values. But how good is "fairly" well? Statistical theory provides several different ways to measure the *goodness of fit*. One of the most commonly used is the χ^2-test.

A fuller description of the test is outside the scope of this document, but the data from the 10,000 trials on the HP 48GX produce a χ^2 value of about 7.322, which puts it in the 39^{st} percentile. Loosely translated, this means that for this theoretical distribution, we may expect a fit this good or better about 39% of the time. Conversely, we may expect a fit this bad or worse about 61% of the time. From this we can conclude that there is no reason to suppose that our experimental results do not match the theory.

The RPL program on the next page is based on the previous one. It is run the same way, but instead of generating a list and an estimate, it produces a graphical display. The expected frequencies are shown as thick bars, the observed frequencies as thin bars.

```
«
 0 11 FOR j
  j
 NEXT
 →LIST SWAP
 { 11 } 0 CON
 1 3 PICK START
  1
  1 10 START
   RAND RAND π * SIN < +
  NEXT
  DUP2 GET 1 + PUT
 NEXT
 SWAP / PICT PURGE
 0 13 XRNG 0 .4 YRNG { # 0h # 0h } PVIEW
 1 11 FOR j
  10 j 1 - COMB 2 π / j 1 - ^ * 1 2 π / - 11 j - ^ *
  i DUP .8 + FOR k
   k OVER R→C k 0 R→C LINE
  .1 STEP
  DROP
  j .3 + DUP .2 + FOR k
   k OVER j GET R→C k 0 R→C TLINE
  .1 STEP
 NEXT
 OBJ→ DROP 11 →LIST * ΣLIST "π=" 20 ROT / +
 PICT { # 0h # 0h } ROT 1 →GROB REPL
 7 FREEZE
»
```