Basic Calculus with the HP49G
Volume 2

By Nick Karagiaouroglou

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Key pressing conventions

→NUM	Right shifted key
DEF	Left shifted key
ENTER	Unshifted key
HALFTAN	Menu key (Soft key)
ASSUME	Select the command from the command catalog of the HP49G or type it in the command line and enter it
R	Alpha shifted key
-F1	Press blue shift, hold it pressed, press F1, and then release both keys
-6	Press red shift, hold it pressed, press key 6, and then release both keys

Before you start working you should set your flags.
Enter the list { #A003008D8103F0h #0h #190101402000028h #0h } and press STOF

Basic Calculus with the HP49G - Volume 2 - Part 4

Hi everybody!

This is part 4 of the Basic Calculus Marathon and it is also the start of volume 2. The first volume of the marathon contains parts 1, 2 and 3, and because it is already so big, I decided to continue in a new separate document. I wonder how many volumes this marathon will have when it is ready.

As we have seen, the HP49G provides very good tools for finding and working with derivatives. We will continue with one of the most typical things that have to do with calculus, namely finding extrema of functions. We begin with functions of a single variable. On the HP49G we can find local maxima and minima in many ways. Let's first choose a function to work with. Go to the EQW and enter $X^3-3\ X^2-9\ X+17$. Store this function in FTEST as we are going to use it more than once. We search for extrema of this function. As you know a function $f(x)$ has a maximum at $x=x_0$ when:

$$\left.\frac{\partial f(x)}{\partial x}\right|_{x=x_0}=0 \text{ and } \left.\frac{\partial^2 f(x)}{\partial x^2}\right|_{x=x_0}<0$$

Similarly a function $f(x)$ has a maximum at $x=x_0$ when:

$$\left.\frac{\partial f(x)}{\partial x}\right|_{x=x_0}=0 \text{ and } \left.\frac{\partial^2 f(x)}{\partial x^2}\right|_{x=x_0}>0$$

That means that we must find the roots of the first derivative of our function, and then plug these roots in the second derivative and check if it is greater or less than 0. Recall FTEST, enter X, press $\boxed{\partial}$ and then **EXPAND** to get $3\ X^2-6\ X-9$. We will need the first derivative later, so store it in FTEST′. (The character "′" is character number 180.) Now we will find the roots of the first derivative. Recall FTEST′, enter X and press **SOLVE**. The HP49G returns $\{X=3\quad X=-1\}$. That means that for $X=3$ and for $X=-1$ the function $X^3-3\ X^2-9\ X+17$ has extrema. Store the list in EXTREMA. Let's find the second derivative of FTEST now. We will differentiate the first derivative for X again. Recall FTEST′, enter X, press $\boxed{\partial}$ and then **EXPAND** to get $6\ X-6$. Store this result in FTEST″. Now we will plug the values in the second derivative. Recall FTEST″ and then the list EXTREMA. Press **SUBST** and expand to get $\{12\quad -12\}$. That means that for $X=3$ the function has a local minimum and for $X=-1$ the function has a local maximum. What are the values of the local maximum and the local minimum? Recall FTEST and EXTREMA and press **SUBST** and **EXPAND**. The HP49G says: $\{-10\quad 32\}$ which means that the function $X^3-3\ X^2-9\ X+17$ at $X=3$ has a local minimum value of -10 and at $X=-1$ a local maximum value of 32.

As you can see, the ability to find maxima and minima analytically strongly depends on the ability to solve analytically the equation:

$$\frac{\partial f(x)}{\partial x}=0$$

If the HP49G can solve that, then you win. If it can't solve that, then you can still win, but you can as well lose. Let's have such an example. We try to find the extrema of the function:

$$\frac{SIN(X)}{X}$$

Enter the above function and store it in FTEST. Take its first derivative:

$$-\frac{SIN(X)-X\ COS(X)}{X^2}$$

and its second derivative:

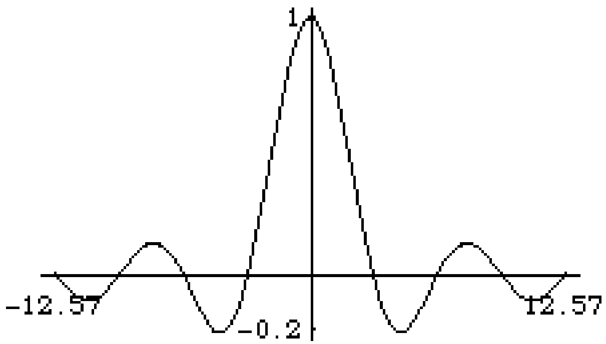
$$-\frac{(X^2-2)\ SIN(X)+2\ X\ COS(X)}{X^3}$$

for X and store them in FTEST′ and FTEST″ respectively, like we did before.

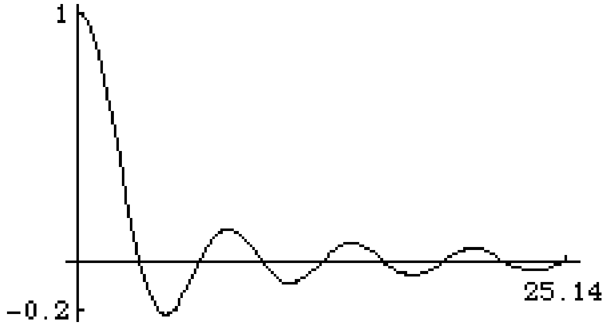
Now, the equation:

$$-\frac{SIN(X)-X\ COS(X)}{X^2}=0$$

can't be solved analytically on the HP49G (or elsewhere). We have to try other techniques to find the roots of this equation. Of course when analytical methods fail, we can try numerical methods, but then, oh then the magic is gone. If we could find some analytic closed solution, then one single formula would give us all solutions of the equation, i.e. all extrema of $\frac{SIN(X)}{X}$. But if we use numerical methods, then we have to apply them over and over again, once for each extremum. Boring! But let's at least take a look at what the HP49G provides in such cases. Store $\frac{SIN(X)}{X}$ in EQ, and do a function plot of it from $X=-12.57$ to $X=12.57$ and with automatic scaling of the Y-view range. As you can see the function is symmetric under coordinates transformation $X \rightarrow -X$, i.e.: $\frac{SIN(X)}{X}=\frac{SIN(-X)}{-X}$.



So we can examine only the function for positive values of X and then extend what we find to negative values of X. When the plot is ready the graphics cursor id positioned on the centre of the plot. Press ↱ and then ▶ to move it to the right edge of the screen leaving its Y-coordinate unchanged. We are going to re-centre the plot at the new graphics cursor position. Press ZOOM then NXT and then CNTR. The plot is redrawn centred at (12.57,.39). As you can see the function has a series of maxima and minima. The maximum at $X=0$ is "easy" to find, since the equation of the first derivative



$$-\frac{SIN(X)-X\ COS(X)}{X^2}$$ is equal to 0 at $X=0$. This maximum is not found analytically but rather empirically through looking at the plot, but nonetheless it is exact and no approximation. Move the graphics cursor somewhere near the X-coordinate of the first minimum. Press FCN and then EXTR. The HP49G starts searching for an extremum of the function. It finds the extremum next to the current X-coordinate of the graphics cursor. When it finds it, it moves the cursor to that point, and displays the X- and Y-coordinates of the extremum at the bottom of the display. A copy of these coordinates is put on stack level 1 as a complex number. Move the cursor to the X-coordinate of the next maximum to the right. Press some menu key to display the menu again, and then EXTR again to find the coordinates of the maximum. Repeat the same procedure for each of the displayed extrema. (That's the boring part.) When done, press CANCEL to return to the stack, which is full of labelled coordinates of extrema. We don't need the Y-coordinates, so let's isolate the X-coordinates of all these points. Press ▲ to go to the interactive stack, and then repeat pressing ▲ until you are at stack

level 7. Press NXT and then →LIST. This puts all objects from the current stack level to stack level 1 in a list. Press CANCEL to exit the interactive stack. Now press RE to get the real part of all the labelled complex numbers in the list, i.e. the X-coordinates of the extrema. Store the list in EXTREMA. Let's examine the list. Since we have to do with a function in which trigonometrics are involved, we "smell" the presence of π. Recall EXTREMA, press π, →NUM and then ÷ to divide all numbers in the list by the numeric value of π. Press ▼ to view the list. Notice that the numbers agree better and better with the formula $\frac{2\ n+3}{2}$, where $n = 0,1,2,$. We already have a first general result from the examination of the extrema of $\frac{SIN(X)}{X}$. For big values of X, the extrema approach $\frac{2\ n+3}{2}$ ı. This is of course no proof, it is only an observation that leads to an assumption about the extrema. Let's make a further test of this assumption. Store the first derivative

$$-\frac{SIN(X)-X\ \ COS(X)}{X^2}$$

in EQ. Press APPS to get the pop-up menu with the built-in applications of the HP49G. Using the arrow keys select 4.Numeric Solver . Press ENTER. Now a new pop-up appears that contains all numeric solvers of the HP49G. The first one is 1.Solve Equation and it is already selected. Press ENTER to go to the SOLVE EQUATION screen. The first derivative is already stored in EQ and so it appears in the input field Eq:. Select the input field X:. Let's use a guess value of $\frac{2\ n+3}{2}$ ı, with $n = 100$. Type 2 100 3 + 2 / π → NUM and press ENTER. The numeric value 318.871654339 is put in the input field X:, and the field Eq: is selected. Select the input field X: again, press NXT and then SOLVE. After a while the HP49G returns 318.868518261 in the input field X:, which is almost what we used as a guess value. It seems that our assumption about the extrema is correct. Press CANCEL to return to the stack which contains the found solution at level 1. Drop the solution. The HP49G has also stored the solution in variable X. Since this is often the variable VX and the CAS doesn't like numeric values stored in VX, purge variable X to avoid problems later. As we see the sequence of X-values that correspond to the extrema of $\frac{SIN(X)}{X}$ resembles the sequence $\frac{2\ n+3}{2}$ ı more and more as n gets greater and greater. Can we use this fact in our investigations? Can we use it to at least get an *analytic approximation* of the extrema, that is better than $\frac{2\ n+3}{2}$ ı? Let's see. Since we smelled that for $X = \frac{2\ n+3}{2}\ \pi$ the function has (almost) extrema and that for growing n we approach the extrema better and better, we can try to make a series expansion at $X = \frac{2\ n+3}{2}\ \pi$, take the derivative of the series, and solve the series for X. Recall FTEST, enter X = X0, then enter 3, and then press SERIES. The HP49G shakes, rattles and rolls and it gasps out a huge list in stack level 2 and $h = X - X0$ in stack level 1. Press SUBST to substitute $X - X0$ for h in all algebraic objects in the list. Enter 3 and press GET to get the third item of the list which is the series expansion. Now press DTAG to get rid of the label. The series expansion that the HP49G returned is of 4th order. Press ▼ to get the series expansion in the EQW for editing. We want a series expansion of 3rd order (didn't I tell you that?), so we are going to delete the 4th order term, i.e. truncate the series the brutal way. In the EQW press ▼ to select the first (4th order) term, and then press DEL. The 4th order term is deleted and the 3rd order term is now selected. Press ENTER to put the edited series expansion on the stack. The expression is now:

$$\frac{\left(3\ X0^2-6\right)\ \mathrm{SIN}(X0)-\left(X0^2-6\ X0\right)\ \mathrm{COS}(X0)}{6\ X0^4}\ (X-X0)^3$$
$$-\frac{\left(X0^2-2\right)\ \mathrm{SIN}(X0)+2\ X0\ \mathrm{COS}(X0)}{2\ X0^3}\ (X-X0)^2$$
$$-\frac{\mathrm{SIN}(X0)+X0\ \mathrm{COS}(X0)}{X0^2}\ (X-X0)+\frac{\mathrm{SIN}(X0)}{X0}$$

Store that in SERTEST. Now, we know that for extrema the equation holds:

$$\left.\frac{\partial f(x)}{\partial x}\right|_{x=x_0}=0$$

where x_0 is the value for which the function $f(x)$ goes through the extremum. The series expansion is:

$$f(x_0)+\frac{1}{1!}\left.\frac{\partial f(x)}{\partial x}\right|_{x=x_0}(x-x_0)+\frac{1}{2!}\left.\frac{\partial^2 f(x)}{\partial x^2}\right|_{x=x_0}(x-x_0)^2+$$

If we use the fact that we did the series expansion at (approximately) the points where the extrema occur, we can stripe off the term

$$-\frac{\mathrm{SIN}(X0)+X0\ \mathrm{COS}(X0)}{X0^2}\ (X-X0)$$

(first derivative) from our series expansion, since it is (approximately) 0.

Recall SERTEST, press ▼ to get it into the EQW, press ▼ and then twice ▶ to select the first order term, and press DEL to let it go. Press ENTER to put the series on the stack. The expression now must be:

$$\frac{\left(3\ X0^2-6\right)\ \mathrm{SIN}(X0)-\left(X0^2-6\ X0\right)\ \mathrm{COS}(X0)}{6\ X0^4}\ (X-X0)^3$$
$$-\frac{\left(X0^2-2\right)\ \mathrm{SIN}(X0)+2\ X0\ \mathrm{COS}(X0)}{2\ X0^3}\ (X-X0)^2+\frac{\mathrm{SIN}(X0)}{X0}$$

Now enter X and press ∂ to get the derivative. You get:

$$\frac{\left(3\ X0^2-6\right)\ \mathrm{SIN}(X0)-\left(X0^2-6\ X0\right)\ \mathrm{COS}(X0)}{6\ X0^4}\ 3\ (X-X0)^2$$
$$-\frac{\left(X0^2-2\right)\ \mathrm{SIN}(X0)+2\ X0\ \mathrm{COS}(X0)}{2\ X0^3}\ 2\ (X-X0)$$

Enter:

$$X0=\frac{2\ n+3}{2}\ \pi$$

and press SUBST to substitute $\frac{2\ n+3}{2}\ \pi$ for $X0$. The expression now is:

$$\frac{\left(3\left(\frac{2\ n+3}{2}\ \pi\right)^2-6\right)\mathrm{SIN}\left(\frac{2\ n+3}{2}\ \pi\right)-\left(\left(\frac{2\ n+3}{2}\ \pi\right)^2-6\ \frac{2\ n+3}{2}\ \pi\right)\mathrm{COS}\left(\frac{2\ n+3}{2}\ \pi\right)}{6\left(\frac{2\ n+3}{2}\ \pi\right)^4}\ 3\left(X-\frac{2\ n+3}{2}\ \pi\right)^2$$
$$-\frac{\left(\left(\frac{2\ n+3}{2}\ \pi\right)^2-2\right)\mathrm{SIN}\left(\frac{2\ n+3}{2}\ \pi\right)+2\ \frac{2\ n+3}{2}\ \pi\ \mathrm{COS}\left(\frac{2\ n+3}{2}\ \pi\right)}{2\left(\frac{2\ n+3}{2}\ \pi\right)^3}\ 2\left(X-\frac{2\ n+3}{2}\ \pi\right)$$

Store this result in DSER1. Recall DSER1 and let's continue. The derivative of the series contains

$$\mathrm{COS}\left(\frac{2\,n+3}{2}\,\pi\right)$$

and

$$\mathrm{SIN}\left(\frac{2\,n+3}{2}\,\pi\right)$$

All cosines of $\frac{2\,n+3}{2}\,\pi$ are equal to 0 for integer values of n. (Should I repeat the story about the missing feature INTEGERASSUME ? ;-)) Enter the list:

$$\left\{\mathrm{COS}\left(\frac{2\,n+3}{2}\,\pi\right)\;0\right\}$$

and press ↑MATCH. Drop the 1 from the stack. The sines of $\frac{2\,n+3}{2}\,\pi$ will be either 1 or -1 for integer values of n. Press ENTER to make a copy of the expression on stack level 1. Enter the list:

$$\left\{\mathrm{SIN}\left(\frac{2\,n+3}{2}\,\pi\right)\;1\right\}$$

press ↑MATCH and drop the 1 again. The expression on stack level 1 is now:

$$\frac{\left(3\left(\frac{2\,n+3}{2}\,\pi\right)^2-6\right)\cdot 1-\left(\left(\frac{2\,n+3}{2}\,\pi\right)^2-6\left(\frac{2\,n+3}{2}\,\pi\right)\right)\cdot 0}{6\left(\frac{2\,n+3}{2}\,\pi\right)^4}\cdot 3\left(X-\frac{2\,n+3}{2}\,\pi\right)^2-\frac{\left(\left(\frac{2\,n+3}{2}\,\pi\right)^2-2\right)\cdot 1+2\left(\frac{2\,n+3}{2}\,\pi\right)\cdot 0}{2\left(\frac{2\,n+3}{2}\,\pi\right)^3}\cdot 2\left(X-\frac{2\,n+3}{2}\,\pi\right)$$

Store the resulting expression in DSERP. Now enter the list:

$$\left\{\mathrm{SIN}\left(\frac{2\,n+3}{2}\,\pi\right)\;-1\right\}$$

press ↑MATCH and drop the 1. The expression on stack level 1 is now:

$$\frac{\left(3\left(\frac{2\,n+3}{2}\,\pi\right)^2-6\right)\cdot -1-\left(\left(\frac{2\,n+3}{2}\,\pi\right)^2-6\left(\frac{2\,n+3}{2}\,\pi\right)\right)\cdot 0}{6\left(\frac{2\,n+3}{2}\,\pi\right)^4}\cdot 3\left(X-\frac{2\,n+3}{2}\,\pi\right)^2-\frac{\left(\left(\frac{2\,n+3}{2}\,\pi\right)^2-2\right)\cdot -1+2\left(\frac{2\,n+3}{2}\,\pi\right)\cdot 0}{2\left(\frac{2\,n+3}{2}\,\pi\right)^3}\cdot 2\left(X-\frac{2\,n+3}{2}\,\pi\right)$$

Store the resulting expression in DSERN. These two expressions are polynomials of second degree in X which we can solve analytically. Actually we only need to solve one of them since DSERP = −DSERN. Recall both DSERP and DSERN on the stack, press + and expand to get 0, which tells us that indeed the one polynomial is the negative of the other. Since the first derivative of some function is 0 at the extrema, recall DSERP, enter X and press SOLVE. The HP49G returns:

$$\left\{X=\frac{(10\,n+15)\,\pi}{6}\quad X=\frac{(2\,n+3)\,\pi}{2}\right\}$$

The first of these two solutions is not useful in this case. The second is exactly our first analytic approximation of the real extrema. So we didn't get a better approximation. But we don't give up. On our way to a better approximation we deleted the first order term of the series expansion because we assumed that it is approximately 0, since the first derivative of the expression vanishes at the extrema. But we didn't have the *exact* extrema. We had only the *approximations* $X0=\frac{2\,n+3}{2}\,\pi$, of which we know that they are *almost* the extrema. Would it help to not delete them? Let's see. Recall SERTEST, enter X and press ∂ to get the derivative:

$$\frac{(3\,X0^2-6)\,SIN(X0)-(X0^2-6\,X0)\,COS(X0)}{6\,X0^4}\,3\,(X-X0)^2-\frac{(X0^2-2)\,SIN(X0)+2\,X0\,COS(X0)}{2\,X0^3}\,2\,(X-X0)-\frac{SIN(X0)+X0\,COS(X0)}{X0^2}$$

Store this in DSER1. Now enter:

$$X0=\frac{2\,n+3}{2}\,\pi$$

and press SUBST to substitute $\frac{2\,n+3}{2}\,\pi$ for X0. Again the sub expressions:

$$COS\left(\frac{2\,n+3}{2}\,\pi\right)$$

are all equal to 0, and the sub expressions:

$$SIN\left(\frac{2\,n+3}{2}\,\pi\right)$$

are all equal to 1 or −1.

Enter the list:

$$\left\{COS\left(\frac{2\,n+3}{2}\,\pi\right)\ 0\right\}$$

and press ↑MATCH. Drop the 1 from the stack. Press ENTER to make a copy of the expression on stack level 1. Enter the list:

$$\left\{SIN\left(\frac{2\,n+3}{2}\,\pi\right)\ 1\right\}$$

press ↑MATCH and drop the 1 again. The expression on stack level 1 is now:

$$\frac{\left(3\left(\frac{2\,n+3}{2}\,\pi\right)^2-6\right)1-\left(\left(\frac{2\,n+3}{2}\,\pi\right)^2-6\left(\frac{2\,n+3}{2}\,\pi\right)\right)0}{6\left(\frac{2\,n+3}{2}\,\pi\right)^4}\,3\left(X-\frac{2\,n+3}{2}\,\pi\right)^2$$
$$-\frac{\left(\left(\frac{2\,n+3}{2}\,\pi\right)^2-2\right)1+2\left(\frac{2\,n+3}{2}\,\pi\right)0}{2\left(\frac{2\,n+3}{2}\,\pi\right)^3}\,2\left(X-\frac{2\,n+3}{2}\,\pi\right)-\frac{1+\left(\frac{2\,n+3}{2}\,\pi\right)0}{\left(\frac{2\,n+3}{2}\,\pi\right)^2}$$

Store the resulting expression in DSERP. Now enter the list:

$$\left\{ \mathrm{SIN}\left(\frac{2\ n+3}{2}\ \pi\right)\quad -1 \right\}$$

press [↑MATCH] and drop the 1. The expression on stack level 1 is now:

$$\frac{\left(3\left(\frac{2\ n+3}{2}\ \pi\right)^2-6\right)\left(1-\left(\left(\frac{2\ n+3}{2}\ \pi\right)^2-6\left(\frac{2\ n+3}{2}\ \pi\right)\right)0\right)}{6\left(\frac{2\ n+3}{2}\ \pi\right)^4}\ 3\left(X-\frac{2\ n+3}{2}\ \pi\right)^2$$

$$-\frac{\left(\left(\frac{2\ n+3}{2}\ \pi\right)^2-2\right)\left(1+2\left(\frac{2\ n+3}{2}\ \pi\right)0\right)}{2\left(\frac{2\ n+3}{2}\ \pi\right)^3}\ 2\left(X-\frac{2\ n+3}{2}\ \pi\right)-\frac{-1+\left(\frac{2\ n+3}{2}\ \pi\right)0}{\left(\frac{2\ n+3}{2}\ \pi\right)^2}$$

Store the resulting expression in DSERN. Again both expressions are negatives of each other, so we only need to solve one of them. Recall DSERP, enter X and press [ZEROS]. The HP49G returns:

$$\left\{ \begin{array}{l} \dfrac{\left((32\ n^3+144\ n^2+216\ n+108)\ \pi^3-(64\ n+96)\ \pi-\pi\sqrt{\left(64\ n^6+576\ n^5+2160\ n^4+4320\ n^3+4860\ n^2+2196\ n+729\right)\pi^4+(128\ n^4+728\ n^3+1728\ n^2+1728\ n+648)\ \pi^2-(512\ n^2+1536\ n+1152)}\right)}{(24\ n^2+72\ n+54)\ \pi^2-48} \\ \dfrac{\left((32\ n^3+144\ n^2+216\ n+108)\ \pi^3-(64\ n+96)\ \pi+\pi\sqrt{\left(64\ n^6+576\ n^5+2160\ n^4+4320\ n^3+4860\ n^2+2196\ n+729\right)\pi^4+(128\ n^4+728\ n^3+1728\ n^2+1728\ n+648)\ \pi^2-(512\ n^2+1536\ n+1152)}\right)}{(24\ n^2+72\ n+54)\ \pi^2-48} \end{array} \right\}$$

Let's see how good these solutions are. Press [OBJ→] to explode the list, and then [◄] to drop the element count. Now the second solution is on stack level 1. Press [ENTER] to make a copy of it. We are going to make a sequence for $n = 0$ to $n = 6$. Enter n, 0, 6, 1, and press [SEQ] to create the sequence. The HP49G returns the sequence in a list after a while. Now we will convert the members of the sequence to numbers. Enter a 1, then the program `<< →NUM >>`, and then press [DOSUBS]. The HP49G converts all expressions in the list to numbers. Are these numbers the values of X for which $\frac{\mathrm{SIN}(X)}{X}$ goes through extrema?

Recall EXTREMA and compare the two lists. The numbers we found are not what we wanted. Press [◄] three times to get rid of the lists and of the solution that we don't need. Now the first solution is on stack level 1. Let's try our luck with it. Press [ENTER] to make a copy of it. Enter n, 0, 6, 1, and press [SEQ] to create the sequence for $n = 0$ to $n = 6$. Let's use MAP instead of DOSUBS to turn the expressions to numbers. Enter the small program `<< →NUM >>`, and then press [MAP]. The HP49G converts all expressions in the list to numbers. Recall EXTREMA and compare the two lists. The numbers we found are almost equal to the numbers in the list which was stored in EXTREMA. And they are indeed good approximations of the real extrema. Press [-] to see that the worst of our analytic approximations differs only about .00091 from the real extremum. Drop the differences list. Store the solution in APREXTR. We have found that the values of X for which $\frac{\mathrm{SIN}(X)}{X}$ goes through extrema can be very well approximated by:

$$X=\frac{\left((32\ n^3+144\ n^2+216\ n+108)\ \pi^3-(64\ n+96)\ \pi-\pi\sqrt{\left(64\ n^6+576\ n^5+2160\ n^4+4320\ n^3+4860\ n^2+2196\ n+729\right)\pi^4+(128\ n^4+728\ n^3+1728\ n^2+1728\ n+648)\ \pi^2-(512\ n^2+1536\ n+1152)}\right)}{(24\ n^2+72\ n+54)\ \pi^2-48}$$

where $n = 0,1,2,\ldots$. Of course the negatives are also such values, i.e.:

$$X = -\frac{\left((32n^3+144n^2+216n+108)\pi^2-(64n+96)\pi-\pi\sqrt{\begin{array}{l}(64n^6+576n^5+2160n^4+4320n^3+4860n^2+2196n+729)\pi^4+\\(128n^4+728n^3+1728n^2+1728n+648)\pi^2-\\(512n^2+1536n+1152)\end{array}}\right)}{(24n^2+72n+54)\pi^2-48}$$

since we examined only the part $\frac{SIN(X)}{X}$ for $X > 0$ and we know that the function $\frac{SIN(X)}{X}$ is symmetric under the coordinates change $X \to -X$. Let's make the formula stored in APREXTR a little bit nicer. Enter $n \geq 0$ and press ASSUME. Drop the assumption from the stack. Recall APREXTR and press ▼ to get it in the EQW. Select the whole square root and press TSIMP. After some centuries the HP49G returns:

$$(2n+3)\sqrt{(16n^4+96n^3+216n^2+216n+81)\pi^4+(32n^2+96n+72)\pi^2-128}$$

Still in the EQW select the whole expression and press DISTRIB and then EVAL. This will convert the expression to:

$$X = \frac{(4n+6)\pi}{3}-\frac{(2n+3)\pi\sqrt{(16n^4+96n^3+216n^2+216n+81)\pi^4+(32n^2+96n+72)\pi^2-128}}{(24n^2+72n+54)\pi^2-48}$$

Select the sub expression $\frac{(4n+6)\pi}{3}$ and press COLLECT and then EVAL. The sub expression is converted to $\frac{(2n+3)\pi 2}{3}$. Select the whole sub expression under the square root, $(16n^4+96n^3+216n^2+216n+81)\pi^4+(32n^2+96n+72)\pi^2-128$ and press again COLLECT and EVAL, to turn it to $((4n^2+12n+9)\pi^2+16)((4n^2+12n+9)\pi^2-8)$. Now select the first sub expression $4n^2+12n+9$ and collect it the same way to $(2n+3)^2$. Repeat the same for the second sub expression $4n^2+12n+9$. Select the whole denominator of the ratio, $(24n^2+72n+54)\pi^2-48$, and collect it to $((4n^2+12n+9)\pi^2-8)\,6$. Collect now the sub expression $4n^2+12n+9$ of the denominator to $(2n+3)^2$. The expression in the EQW must be now:

$$X = \frac{(2n+3)\pi 2}{3}-\frac{(2n+3)\pi\sqrt{((2n+3)^2\pi^2+16)((2n+3)^2\pi^2-8)}}{((2n+3)^2\pi^2-8)\,6}$$

Press ENTER to put the edited expression to the stack, and store it in APREXTR. Now we are going to try to understand why the expression $\frac{2n+3}{2}\pi$ is a good approximation for big values of n, and what it has to do with:

$$X = \frac{(2\ n+3)\ \pi 2}{3} - \frac{(2\ n+3)\ \pi\sqrt{\left((2\ n+3)^2\ \pi^2+16\right)\left((2\ n+3)^2\ \pi^2-8\right)}}{\left((2\ n+3)^2\ \pi^2-8\right)\ 6}$$

Recall APREXTR and take it to the EQW. When we move to big values of n, then we have:

$$(2\ n+3)^2\ \pi^2+16 \approx (2\ n+3)^2\ \pi^2$$

and also:

$$(2\ n+3)^2\ \pi^2-8 \approx (2\ n+3)^2\ \pi^2$$

That means:

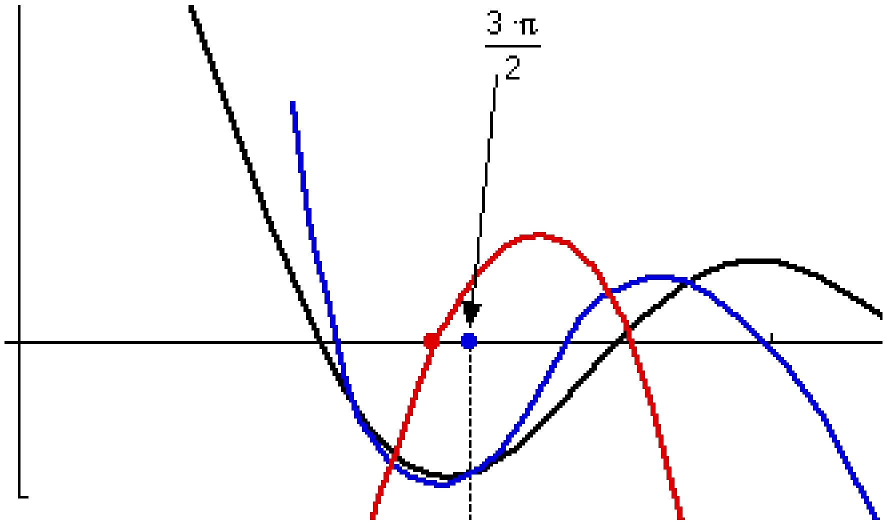
$$X = \frac{(2\ n+3)\ \pi 2}{3} - \frac{(2\ n+3)\ \pi\sqrt{\left((2\ n+3)^2\ \pi^2+16\right)\left((2\ n+3)^2\ \pi^2-8\right)}}{\left((2\ n+3)^2\ \pi^2-8\right)\ 6} \approx$$

$$\frac{(2\ n+3)\ \pi\, 2}{3} - \frac{(2\ n+3)\ \pi\sqrt{(2\ n+3)^2\ \pi^2\ (2\ n+3)^2\ \pi^2}}{(2\ n+3)^2\ \pi^2\ 6} =$$

$$\frac{(2\ n+3)\ \pi\, 2}{3} - \frac{(2\ n+3)\ \pi}{6} = \frac{(2\ n+3)\ \pi}{2}$$

Our approximated formula is good enough to create very good numeric results and to explain why bigger values of n result in X-values for the extrema, which behave more and more like $\frac{2\ n+3}{2}\pi$. The HP49G was a great helper in our quest for an analytic approximation of the extrema of $\frac{\mathrm{SIN}(X)}{X}$. Now we can calculate numeric values of the extrema not only for big but for small values of n with a maximum difference of less than $1E-3$ from the real values, *and* we can examine the behaviour of the extrema further, *and* we can create a user defined function or program that calculates the extrema for any n instead of working interactively in the graphics environment, *and* we can use this user defined function or program in other programs. For those who still prefer the HP48: Do the same! (Without support from Mathematica of course. He, he, you are going to wait a loooooong time, if you ever going to reach the end.) The HP49G *is* the flagship. It is sometimes a strange flagship with many quirks and a peculiar idiosyncrasy, but it is the flagship. Compared to it the HP48 is more sort of fisherman's boat. Period!

Before we go further, we repeat how to use the technique using the picture below. We approximate the original function with a polynomial of degree 3 at the point in the neighbourhood of which we have extrema (blue curve, blue point). We used the parameter n to distinguish between the different extrema. Then we differentiate that polynomial. We get a quadratic polynomial (red curve) and we solve this instead of the derivative of the original function. One of its solutions is a better analytic approximation of the extrema (red point).



We continue with yet another demonstration of the superiority of this machine compared to the fisherman's boats. First enter n and press

UNASSUME to remove all assumptions about n. Press ◄ to drop 'n' from the stack. We are going to use Newton's method to get analytic closed approximations of the extrema fo the function $a\ X^4 + b\ X^3 + X$. I include in this marathon Aaron's excellent posting about Newton's Method. I only replaced the text formulae and the keys with graphics for better lookings. Here we go:

——————————— Aaron starts here ———————————

The first of teaching aids for Calculus with the HP-49G and the TI-89. This topic will cover the Newton-Raphson recursive algorithm, commonly known as Newton's Method. Given the fact that we already know how to calculate derivatives and find the equations of tangent lines to curves at a given point, we will move on. To set the stage for a problem, suppose a used car
dealer offers to sell you a car for $18,000 or for payments of $375 per month for five years. You would like to know what monthly interest rate the dealer is, in effect, charging you. To find the answer, you have to solve the equation (for our time here we will not discuss how we come up with the equation. Accept it right now for no better reason than authority):

$$48\ x\ (1+x)^{60} - (1+x)^{60} + 1 = 0$$

How would you approach solving it? (Note for a quadratic $a\ x^2 + b\ x + c$, there is a well known formula to find the roots called the quadratic equation, 3rd and 4th degree equations get much more complicated and there are formulas to find the roots there, but if $f(x)$ is a polynomial of degree 5 or higher, there is no such formula to find exact roots.) We could graph the function in our HP-49 or TI-89, set the viewing rectangle to $x \to \{0,.012\}$, $y \to \{-.05,.15\}$, and then use the trace function to approximate the root between $.007$ and $.008$. Zooming in repeatedly, we could find correct to nine decimal places that the root is $.007628603$. But this is tiresome, redundant, and takes a great deal of time. We could use the Solve( ) command in our calculators to find the approx solution as well. But how does the calculator find the root? They use a variety of methods, but the most commonly used method is Newton's Method. Now what is Newton's Method? Lets discuss it detail:

Suppose you have a curve that has a root R and suppose R is not known. To find R, we take a known value close to R and call it x_1. Then we locate the y-value on the curve so that we have a point $(x_1, f(x_1))$. Then calculate the tangent line at that point and sketch it such that the tangent line crosses the x axis. That root where the tangent line crossed we will call x_2. Then find the y-value of x_2 and repeat the process. What you will find is in effect x_2, x_3, etc will get closer and closer to your R root (there are cases where this will fail, we will discuss these later). In fact, you only need to find about x_5 or x_6 to be correct to 6-8 decimal places! To find a formula for x_2 in terms of x_1, we use the fact that the slope of L is f'9x10, so its equation is:

$y - f(x_1) = f'(x_1)\ (x - x_1)$; where $f'(x_1)$ is the derivative of $f(x_1)$.

Since the x-intercept of L is x_2, we set $y = 0$ and obtain;

$$0 - f(x_1) = f'(x_1)\ (x_2 - x_1).$$

If $f'(x_1) \neq 0$, we can solve this equation for x_2:

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

We use x_2 as a second approximation to R:

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)}$$

If we keep repeating this process, we obtain a sequence of approximations $x_1, x_2, x_3, x_4,$ In general, if the nth approximation is x_n and $f'(x_n) \neq 0$, then the next approximation is given by:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}.$$

If the numbers x_n become closer and closer to R as n becomes large, then we say that the sequence converges to R and we write:

$$\lim_{n\to\infty} x_n = R$$

Although the sequence fo successive approximations converges to the desired root for some functions, in other circumstances the sequence may not converge. This is likely to be the case when $f'(x_1)$ is close to 0. It might even happen that an approximation falls outside the domain of f. THEN NEWTONS METHOD FAILS AND A BETTER INITIAL APPROXIMATION X1 SHOULD BE CHOSEN.

Now how would you use this with the HP-49 or the TI-89? Well, each calculator goes about it differently but the idea is the same. Suppose we want to find the root of $x^3 + x + 1 = 0$. Let's make our first guess (x_1) be -1. $f'(x) = 3\ x^2 + 1$, so our equation would be:

$$(-1) - \frac{(-1)^3 + (-1) + 1}{3\ (-1)^2 + 1}.$$

Which yields $-.75$. Putting $.75$ in x_2 and re-evaluating gives us $-.686046511628$. Put our answer now in for x_3 and evaluate again, and we get $-.682339582597$. One more time yields $-.682327803947$, and a last evaluation gives us $-.682327803828$. (Note: Notice how on our 2nd and 3rd evaluations $.68$ is repeated, and on our 3rd and 4th $.6823$ is repeated and on our 4th and 5th $-.682327803$ is repeated? Newton brought to light something interesting when coming up with his recursive formula. For each evaluation after x_2, your accuracy will double. Notice we have two decimal places of accuracy by our 3rd evaluation, 4 by our fourth, and 9 by our fifth. Newton's Method is a GREAT way to get accurate in a hurry. Chances are by our 6th evaluation, we would be accurate to 18 decimal places!)

Now those of you with the HP-49, you can program it so each time you hit enter, your answer will be displayed. To program the algorithm, do the following (assuming you are in RPN mode): Place your first equation ($f(x)$) on the stack. Press Y, then STO. Now put the derivative of your first equation ($f'(x)$) on the stack. Press Z then STO. Press your first initial guess on the stack (in our previous example, it would be -1), press X then STO. Now for the program. Key in the following, then press enter to place it on the stack:

```
<< X Y Z / - →NUM { X } PURGE X STO X >>
```

Now that it is on the stack, press ALPHA twice, type NEWT, the STO. If you press the VAR button you will notice your variables X, Y, Z, and NEWT above their respective soft keys. Now every time your press NEWT, you will get a numerical value closer and closer to your root. On the TI-89, the idea is the same. Press ◆ then F1. This brings you to the $y =$ screen. Type your first equation in $y1$, and your second equation in $y2$. Then press the home button. For your program here, press your first initial guess (ours was -1) the press STO→, press X, then ENTER. Here you assigned a value to x, Then in this sequence, press the following:

```
X - Y1(X) / Y2(X) STO→ X ENTER
```

Each time you press ENTER, you will get closer and closer to the root you are seeking.

I know this was a lengthy post, so your comments are appreciated. Any questions, feel free to email me, or respond to the post. Thanks!

-Aaron

——————— Aaron ends here ———————

Comments? What comments? There can be no other comments but a big fat "Thank you" for such an excellent posting. I wish there would be more postings like this, which not only illustrate so nicely the usage of our tools, but which at the same time also show that there are no "dark sides", "light sides" and other sides whatsoever. Hopefully we will be able to converge at the end, and leave the stupid prejudice-patterns behind us.

The above posting of Aaron describes a method for finding analytic approximations of extrema of functions when the HP49G can't find the roots of the first derivative of some function analytically. We use again one example to demonstrate the method. We try to find the extrema of the function $\mathrm{SIN}(3\ X)+e^{-X}$ from 0 to $+\infty$. This function consists of two parts, one of which is an exponential. The exponential part rapidly approaches 0 when X approaches $+\infty$. So we expect that the function is essentially equal to $\mathrm{SIN}(3\ X)$ for big values of X.

The extrema of this function are $\frac{n\ \pi}{3}+\frac{\pi}{6}$. This will be our "guess" value for finding the extrema using Newton's method. We will use this method to find analytic approximations of the extrema of $\mathrm{SIN}(3\ X)+e^{-X}$. That means that we will use the method for finding the roots of the first derivative of the function. Enter $\mathrm{SIN}(3\ X)+e^{-X}$, then enter X, and press $\boxed{\partial}$ to get the first derivative, $3\ \mathrm{COS}(3\ X)+e^{-X}\ {-1}$. This will be the function for which we want to find the roots. The roots of this function are the extrema of the original function. Store the function in FTEST. Recall FTEST, enter X, and press $\boxed{\partial}$ to get the derivative of the derivative, $3\ (-3\ \mathrm{SIN}(3\ X))+e^{-X}\ {-1}\ {-1}$. Store it in FTEST'. Now we take a closer look to Newton's method again. As Aaron said, if we start at some "guess" value x_0 for the root, the first approximation that this method gives us, is:

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

The second approximation is:

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

If we substitute:

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

in formula for the second approximation, we get:

$$x_2 = x_0 - \frac{f(x_0)}{f'(x_0)} - \frac{f\left(x_0 - \frac{f(x_0)}{f'(x_0)}\right)}{f'\left(x_0 - \frac{f(x_0)}{f'(x_0)}\right)}$$

Let's try first to find out how good the first approximation is. Enter $F(X)$, recall FTEST, and press $\boxed{=}$. Now you have $F(X)=3\ \mathrm{COS}(3\ X)+e^{-X}\ {-1}$ on stack level 1. Press $\boxed{\mathrm{DEF}}$ to make the user defined function F. Now enter $F'(X)$, recall FTEST', press $\boxed{=}$ and then $\boxed{\mathrm{DEF}}$ to make the user defined function F'. Enter:

$$\mathrm{APPREX}(X)=X-\frac{F(X)}{F'(X)}$$

and press **DEF** to make the user defined function APPREX which returns the approximations of roots using Newton's method. Now, enter our "guess" for the extrema, $\frac{n\cdot\pi}{3}+\frac{\pi}{6}$. Press **APPREX** to get:

$$\frac{n\cdot\pi}{3}+\frac{\pi}{6}+\frac{3\cdot\cos\left(3\cdot\left(\frac{n\cdot\pi}{3}+\frac{\pi}{6}\right)\right)+e^{-\left(\frac{n\cdot\pi}{3}+\frac{\pi}{6}\right)}-1}{3\cdot\left(-3\cdot\sin\left(3\cdot\left(\frac{n\cdot\pi}{3}+\frac{\pi}{6}\right)\right)\right)+e^{-\left(\frac{n\cdot\pi}{3}+\frac{\pi}{6}\right)}\cdot -1\cdot -1}$$

Press **▼** to get the expression in the EQW for a little bit editing. Select the sub expression:

$$3\cdot\left(\frac{n\cdot\pi}{3}+\frac{\pi}{6}\right)$$

in the COS function and expand it to get:

$$\frac{(2\cdot n+1)\cdot\pi}{2}.$$

Now select the sub expression

$$\cos\left(\frac{(2\cdot n+1)\cdot\pi}{2}\right)$$

and press **TEXPAND**. The sub expression is converted to:

$$0\cdot\cos(n\cdot\pi)-1\cdot\sin(n\cdot\pi)$$

Since $\sin(n\cdot\pi)=0$ for integer n, the whole sub expression $0\cdot\cos(n\cdot\pi)-1\cdot\sin(n\cdot\pi)$ is equal to 0. While the whole sub expression is selected, press **0** to replace the whole sub expression with 0. Now select the sub expression:

$$3\cdot\left(\frac{n\cdot\pi}{3}+\frac{\pi}{6}\right)$$

in the SIN function, expand it, select the sub expression

$$\sin\left(\frac{(2\cdot n+1)\cdot\pi}{2}\right)$$

and press **TEXPAND** and then **EXPAND**. The sub expression is converted to $\cos(n\cdot\pi)$. Press **ENTER** to put the edited expression on the stack, which now should be:

$$\frac{n\cdot\pi}{3}+\frac{\pi}{6}+\frac{3\cdot 0+\frac{1}{e^{\frac{n\cdot\pi}{3}+\frac{\pi}{6}}}-1}{-(3\cdot 3\cdot\cos(n\cdot\pi))+\frac{1}{e^{\frac{n\cdot\pi}{3}+\frac{\pi}{6}}}\cdot -1 -1}$$

The expression $\cos(n\cdot\pi)$ is either 1 or -1 for integer n. Press **ENTER** to make a copy of the expression, enter $\{\cos(n\cdot\pi)\ \ -1\}$, and press **↑MATCH** to convert the expression to:

$$\frac{n\cdot\pi}{3}+\frac{\pi}{6}+\frac{3\cdot 0+\frac{1}{e^{\frac{n\cdot\pi}{3}+\frac{\pi}{6}}}-1}{-(3\cdot 3\cdot -1)+\frac{1}{e^{\frac{n\cdot\pi}{3}+\frac{\pi}{6}}}\cdot -1\cdot -1}$$

Press [▼], select the sub expression:

$$\frac{3\ 0+\frac{1}{e^{\frac{n}{3}\pi+\frac{\pi}{6}}}\ \ \cancel{1}}{-(3\ 3\ \ \cancel{1})+\frac{1}{e^{\frac{n}{3}\pi+\frac{\pi}{6}}}\ \ \cancel{1}\ \ \cancel{1}}$$

and expand to convert it to:

$$\frac{1}{9\ e^{\frac{(2\ n+1)\ \pi}{6}}+1}$$

Press [ENTER]. The expression is now:

$$\frac{n}{3}\pi+\frac{\pi}{6}+\frac{1}{9\ e^{\frac{(2\ n+1)\ \pi}{6}}+1}$$

Store it in EXNEG. Enter {COS(n)π 1} and press [↑MATCH] again. Now you get:

$$\frac{n}{3}\pi+\frac{\pi}{6}+\frac{3\ 0+\frac{1}{e^{\frac{n}{3}\pi+\frac{\pi}{6}}}\ \ \cancel{1}}{-(3\ 3\ 1)+\frac{1}{e^{\frac{n}{3}\pi+\frac{\pi}{6}}}\ \ \cancel{1}\ \ \cancel{1}}$$

on stack level 2, and a 1 on stack level 1. Drop the 1. Press [▼], select the sub expression:

$$\frac{3\ 0+\frac{1}{e^{\frac{n}{3}\pi+\frac{\pi}{6}}}\ \ \cancel{1}}{-(3\ 3\ 1)+\frac{1}{e^{\frac{n}{3}\pi+\frac{\pi}{6}}}\ \ \cancel{1}\ \ \cancel{1}}$$

and expand to convert it to:

$$\frac{-1}{9\ e^{\frac{(2\ n+1)\ \pi}{6}}-1}$$

Press [ENTER]. The expression is now:

$$\frac{n}{3}\pi+\frac{\pi}{6}+\frac{-1}{9\ e^{\frac{(2\ n+1)\ \pi}{6}}-1}$$

Store it in EXPOS.

Let's test how good the extrema are represented by EXPOS or EXNEG. Recall EXNEG, enter n, 0, 5, 1, and press [SEQ] to make a sequence of the approximated extrema for $n = 0,1,2,3,4,5$. When the HP49G is ready, enter the program << →NUM >>, and then press [MAP], to convert all expressions in the list to numbers. The list contains now numbers that are very close to the actual extrema of the original function. The expression:

$$\frac{n}{3}\pi+\frac{\pi}{6}+\frac{1}{9\ e^{\frac{(2\ n+1)\ \pi}{6}}+1}$$

is already a very good analytic approximation of the extrema of $SIN(3\ X)+e^{-X}$. You can check this by finding the extrema interactively in the plotting environment, as shown on page 1-2 of this marathon. And it is not only that it gives good numerical results. It also allows us to understand how the extrema behave. In the formula:

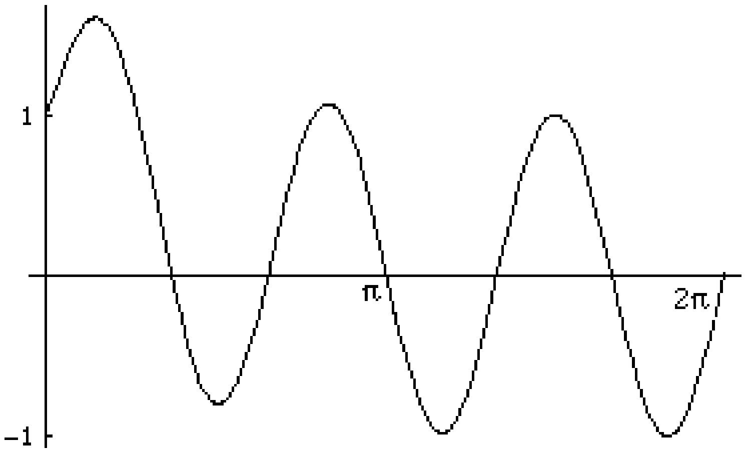
$$\frac{n\ \pi}{3}+\frac{\pi}{6}+\frac{1}{9\ e^{\frac{(2\ n+1)\ \pi}{6}}+1}$$

the term:

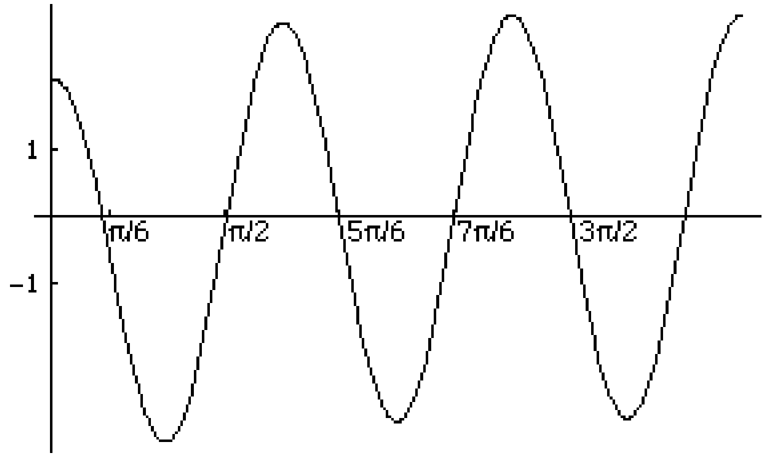
$$\frac{1}{9\ e^{\frac{(2\ n+1)\ \pi}{6}}+1}$$

rapidly approaches 0 as n gets greater. This means that the extrema are represented better and better by $\frac{n\ \pi}{3}+\frac{\pi}{6}$ for big values on n.

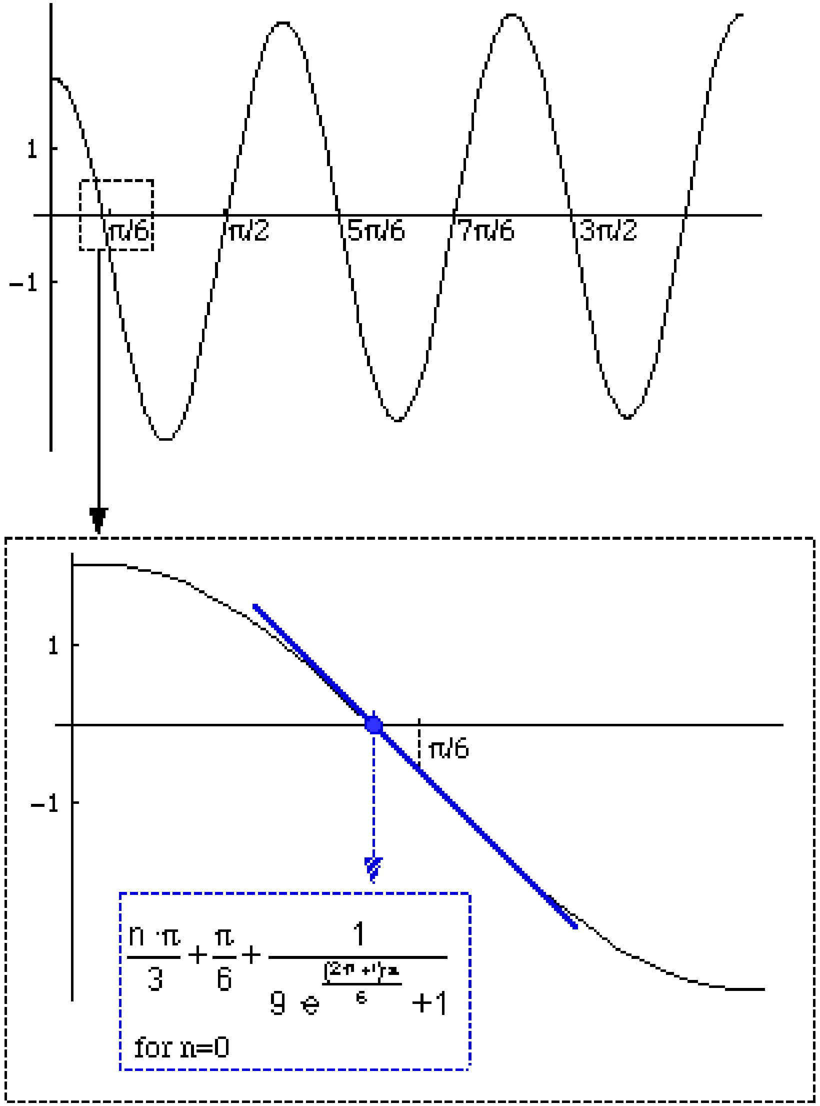
Let's consider the method by means of a picture. First of all we have the function $\mathrm{SIN}(3\ X)+e^{-X}$, whose graph we see below. We find



the derivative of this function, $3\ \mathrm{COS}(3\ X)-e^{-X}$, whose graph we see below. The roots of the first derivative are the extrema of the original



function. Using Newton's method, we find the tangent at our "guess" values for the roots, and then find where the tangent cuts the X-axis. This point is the (first) analytic approximation of one of the roots of $3\ \mathrm{COS}(3\ X)-e^{-X}$, i.e. the (first) analytic approximation of one of the extrema of $\mathrm{SIN}(3\ X)+e^{-X}$. The picture on the next page illustrates this. Of course we can proceed and use the first analytic approximation to find out a second (better) analytic approximation. But in this case it is enough to have the first one, since the second doesn't bring much better numeric values and it makes understanding harder, because it is much more complicated.



The question the remains is: what if the first and the second derivative of the function are both 0 at the $x=\xi$ (extremum)? How can we find out the extremum then? In this case we use higher derivatives. If a function $f(x)$ is differentiable at least n-times ($n \geq 2$), then if the function has an extremum if n is even, and:

$$\left.\frac{\partial f(x)}{\partial x}\right|_{x=\xi} = \left.\frac{\partial^2 f(x)}{\partial x^2}\right|_{x=\xi} = \quad = \left.\frac{\partial^{n-1} f(x)}{\partial x^{n-1}}\right|_{x=\xi} = 0$$

and:

$$\left.\frac{\partial^n f(x)}{\partial x^n}\right|_{x=\xi} \neq 0$$

In case $\left.\frac{\partial^n f(x)}{\partial x^n}\right|_{x=\xi} > 0$ we have a minimum. In case $\left.\frac{\partial^n f(x)}{\partial x^n}\right|_{x=\xi} < 0$ we have a maximum.

Consider for example the function X^4. The first derivative is $4\ X^3$. It has a root at $X=0$, so we assume a possible extremum there. The second derivative is $12\ X^2$, and it is also equal to 0 at $X=0$. So we must use higher derivatives. The third and the fourth derivatives are $24\ X$ and 24 respectively. The function is differentiable four times. So we have $n=4$. For the derivatives we have:

$$\left.\frac{\partial X^4}{\partial X}\right|_{x=0} = \left.\frac{\partial^2 X^4}{\partial X^2}\right|_{x=0} = \left.\frac{\partial^3 X^4}{\partial X^3}\right|_{x=0} = 0$$

and:

$$\left.\frac{\partial^4 X^4}{\partial X}\right|_{x=0} = 24 > 0$$

which means that the function X^4 has a minimum at $X = 0$.

On the HP49G we don't have to follow the above cumbersome procedure for monovariate functions. The command TABVAR is what we need. It takes a monovariate function and returns the function itself at stack level 3, a list representing the variation table of the function on stack level 2, and a graphics object on stack level 1, which contains the same information like the list on stack level 2, but in graphics format. The variable of the function must be your current VX. Let's have one example. If your current VX is not X, then enter X and press STOVX. Enter $X^3 - X$ and press TABVAR. The HP49G returns the function itself, $X^3 - X$, on stack level 3, the list:

$$\left\{\left\{-\infty \quad + \quad -\frac{\sqrt{3}}{3} \quad - \quad \frac{\sqrt{3}}{3} \quad + \quad +\infty\right\} \left\{-\infty \quad \uparrow \quad \frac{2\sqrt{3}}{9} \quad \downarrow \quad -\frac{2\sqrt{3}}{9} \quad \uparrow \quad +\infty\right\}\right\}$$

on stack level 2, and a graphics object on stack level 1, which looks like:

$$F := \left(X^3 - X\right)$$

$$F' := \left(3\ X^2 - 1\right)$$

$$\to : \frac{\left(3\ X + \sqrt{3}\right)\left(3\ X - \sqrt{3}\right)}{3}$$

Variation table:

$$\begin{bmatrix} -\infty & + & -\frac{\sqrt{3}}{3} & - & \frac{\sqrt{3}}{3} & + & +\infty & X \\ -\infty & \uparrow & \frac{2\sqrt{3}}{3} & \downarrow & -\frac{2\sqrt{3}}{3} & \uparrow & +\infty & F \end{bmatrix}$$

The results mean:

The function was $X^3 - X$ and its first derivative was $3\ X^2 - 1$. The HP49G found the roots of the first derivative. In order to do so it had to factor the first derivative. It found:

$$\frac{\left(3\ X + \sqrt{3}\right)\left(3\ X - \sqrt{3}\right)}{3}$$

The roots of this expression were:

$-\frac{\sqrt{3}}{3}$ and $\frac{\sqrt{3}}{3}$

These are the values of X for which the function $X^3 - X$ goes through extrema. The matrix on the graphics object and the list on stack level 2 say more about these extremal values. Let's take a look at the matrix:

$$\begin{bmatrix} -\infty & + & -\frac{\sqrt{3}}{3} & - & \frac{\sqrt{3}}{3} & + & +\infty & X \\ -\infty & \uparrow & \frac{2\sqrt{3}}{3} & \downarrow & -\frac{2\sqrt{3}}{3} & \uparrow & +\infty & F \end{bmatrix}$$

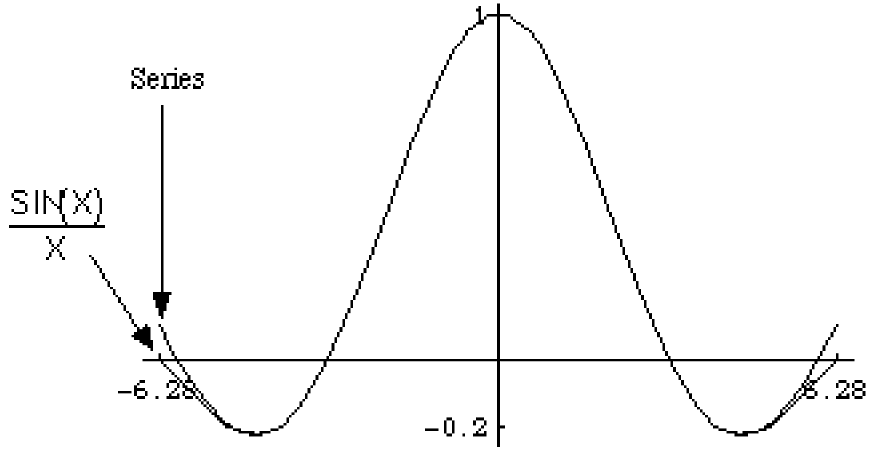
We see that when X comes from $-\infty$, the function increases from $-\infty$ until X has the value $-\frac{\sqrt{3}}{3}$. The function at this point has a local maximum, which is $F(X) = \frac{2\sqrt{3}}{3}$. Then, after $X = -\frac{\sqrt{3}}{3}$, the function decreases until $X = \frac{\sqrt{3}}{3}$, where the function has a local minimum, which is $F(X) = -\frac{2\sqrt{3}}{3}$. After $X = \frac{\sqrt{3}}{3}$ the function increases again to $+\infty$. The same information as in the matrix is contained in the list on stack level 2. The results of the command TABVAR depend on the capability of the HP49G to solve $F'(X) = 0$. If you for example enter $\frac{SIN(X)}{X}$ and press TABVAR, the HP49G will complain Not reducible to a rational expression because in can't solve:

$$\frac{\partial \frac{SIN(X)}{X}}{\partial X} = 0$$

In such cases we must try other methods, like those described in the previous pages. If we are interested for a specific interval of X values, then we can try to make a series expansion around the centre of the interval and work with the expansion instead of the function itself. The results will of course be only an approximation. Suppose for example that we want the extrema of $\frac{SIN(X)}{X}$ in the interval from -2π to 2π. We can use the series expansion of the function at $X = 0$. Of course the question is what order should we choose for an adequate representation of the function by the series. Plotting the function and the series together can help to answer this question. Enter $\frac{SIN(X)}{X}$ and press ENTER to make a copy of the function. Enter $X = 0$ (the point at which we make the series expansion), and then 12 (the order). Press SERIES. The HP49G returns a list on stack level 2 and the equation $h = X$ on stack level 1. Press SUBST to substitute $h = X$ in all expressions contained in the list. The series is the third element of the list, so enter 3 and press GET to extract it from the list. Press DTAG to get rid of the label. Now you have the polynomial

$$\frac{1}{6227020800}X^{12} + \frac{-1}{39916800}X^{10} + \frac{1}{362880}X^{8} + \frac{-1}{5040}X^{6} + \frac{1}{120}X^{4} + \frac{-1}{6}X^{2} + 1$$

on stack level 1. Enter 2 and press →LIST to make a list that contains the original function and the series. Press STEQ to store the list in EQ. If you now plot the two functions from -2π to 2π you see that the series represents the function quite well, especially at the extrema which are of interest for us. Recall EQ, enter 2 and press GET to extract the



series from the list. Before you do anything else set flag -109 to allow numerical factorisation. This is important because the polynomial can't be factored symbolically and so numerical methods must be involved in order to find its roots and extrema. When this flag is set the HP49G will automatically use numerical methods for factorisation, if symbolical methods fail. Now press TABVAR. The HP49G needs a couple of seconds to answer, so be patient. When it finishes a big graphics object is on stack level 1. Press ▼ to view it. The graphics object is displayed centred on the screen, so you must move around to see the information you want. Press and then ◀ to activate scroll mode. Now press and then ◀ to go to the left of the graphics object. Press and then ▼ to go to the bottom. At the bottom of the screen you see the left part of the variation table. It says that when X goes to

−∞ the polynomial goes to 1.6059 E490 . This number should have been exactly ∞, but the HP49G returns this number because of the numerical methods that have been used. This part of the variation table doesn't apply to the function itself, since the polynomial represents the function well only in the interval from −2 ı to 2 ı. Press ▶ and hold it pressed to scroll to the right. You see that the polynomial has a minimum at X = −4.48015167546. The polynomial has the the value −.216280730319. This is in very good agreement with the results that we have obtained previously for the function $\frac{\mathrm{SIN}(X)}{X}$. If you move a little bit more to the right you will see that the HP49G found that the polynomial (and so also the function) has a maximum at X = 0. Another minimum occurs at X = 4.48015167546. We see that an adequate series expansion can help us examine the behaviour of a function in some interval. The results of TABVAR have to be examined thoroughly though. There might be points where the series behaves very differently.

Notice also what TABVAR has done apart from returning the results. It has switched to approximate mode because it used numerical factorisation. Press ↱ ENTER to return to exact mode. Also, it has put the series expansion in EQ overwriting its old contents. If you have some other expressions in EQ that you want to keep. then you should store them in some other variable before using TABVAR . The command changes the contents of PPAR too. It alters the viewing range parameters so that the examined function can be plotted including the extrema that TABVAR has found. Again, if you would like to keep your own plot settings, you should save the contents of PPAR in some other variable before using TABVAR .

Another problem that TABVAR has is that it can't handle functions that contain other additional parameters except the variable VX. If you enter $a^2 \;\; X^2$ and press TABVAR, then the HP48G will complain Parameters not allowed. (In this case TABVAR fortunately doesn't change EQ and PPAR .) In the case of $a^2 \;\; X^2$ it should have been easy for TABVAR to return the minimum that the function has at X = 0. And there are yet additional problems, even for monovariate functions without any parameters. Enter $\mathrm{COSH}(X^2 - 1)$ and press TABVAR. After some seconds the HP49G complains: Not reducible to a rational expression. Yes, my machine, but you *are* able to convert hyperbolics to exponentials. If I press EXPLN to convert to exponentials, you answer correctly:

$$\frac{e^{X^2-1} + \frac{1}{e^{X^2-1}}}{2}$$

Pressing TABVAR now still results in Not reducible to a rational expression. If I press EXPAND you answer:

$$\frac{\left(e^{X^2-1}\right)^2 + 1}{2\ e^{X^2-1}}$$

Pressing TABVAR now still makes you complain Not reducible to a rational expression. But you *are* able to find the extrema of the function, my machine! If I differentiate the expression

$$\frac{\left(e^{X^2-1}\right)^2 + 1}{2\ e^{X^2-1}}$$

for X and then expand it, you answer:

$$\frac{X\left(e^{X^2-1}\right)^2 - X}{e^{X^2-1}}$$

If I solve this for X then you say:

$$\left\{X=-1 \quad X=1 \quad X=-\sqrt{LN\left(e^1\right)} \quad \sqrt{LN\left(e^1\right)} \quad X=0\right\}$$

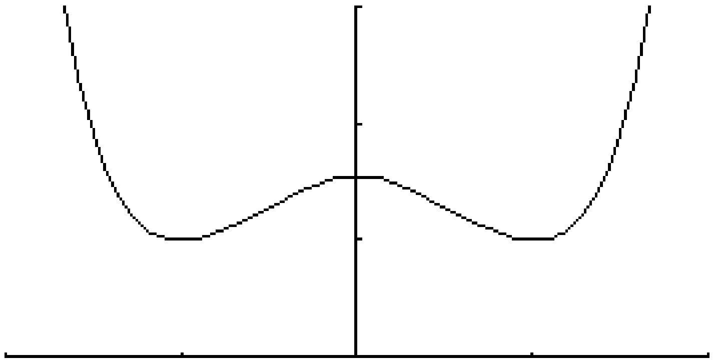
which shows that you should find the extrema of the function at $X=-1$, $X=0$, and $X=1$ when I pressed TABVAR. Since TABVAR seems to be unwilling to do what you can do, my machine, I will program you. Let's see if we can produce supplementary code that we can use when TABVAR doesn't want to tell us the truth. Consider the program:

```
<<
  {} {}
  →  f v f´ sols
  <<
    'f´' f v ∂ STO+          @Find 1st. der., add to f´
    HEAD v                   @Prepare for SOLVE
    IFERR                    @If SOLVE errors out
      SOLVE
    THEN                     @then clean up and return
      DROP2 f v              @user input
      "FINDEX Error:         @and mimic system errors
Can't solve f´=0
Reason:
"
      ERRM + 1 DISP
      1200 .08 BEEP
      3 FREEZE
    ELSE                     @else (SOLVE worked)
      IF                     @If solution was empty list
        DUP {} SAME
      THEN                   @then clean up and return
        DROP f v             @user input
        "FINDEX Error:
No solutions of f´=0
were found"
        1 DISP 1200 .08 BEEP
        3 FREEZE             @Mimic system errors
      ELSE                   @else (solutions weren't {})
        IF                   @If solutions weren't a list
          DUP TYPE 5 ≠
        THEN                 @then convert them to a list
          1  →LIST
        END
        'sols' STO
        f´ v ∂               @find 2nd derivative
        DUP 'f´' SWAP STO+   @add to list f´
        HEAD sols SUBST      @subst. solut. in 1st. der.
        sols 2               @Do for all solutions
        <<
          CASE
              OVER 0 ==      @in case 2nd. der equals 0
            THEN
              "HIGHER"       @return "HIGHER"
            END
              OVER 0 ≥ NOT   @in case 2nd. der. < 0
            THEN
              "MAX"          @return "MAX"
            END
              OVER 0 ≤ NOT   @in case 2nd. der > 0
            THEN
              "MIN"          @return "MIN"
            END
          END
          →TAG               @Label solution
          2  →LIST           @Wrap function value and sol.
        >>                   @in a list
        DOLIST
        1                    @Do for all sub lists
        <<
          OBJ→  DROP         @Explode
          2 f´ 2 GET
          →  derV sol n derF
          <<
            IF               @If solution is labeled
              sol OBJ→  NIP  @with "HIGHER"
```

```
      "HIGHER" SAME
    THEN               @then keep on finding
      WHILE            @higher derivatives
        derV 0 ==
      REPEAT
        derF v ∂
        'derF' STO
        'n' 1 STO+     @keep track of deriv. order
        derF sol
        SUBST          @Substitute solutions in
        'derV' STO     @higher derivative
      END
      IF
        n 2 MOD NOT    @If der. order is even
      THEN
        CASE           @in case higher der. < 0
            derV 0
            ≥ NOT
          THEN         @return "MAX"
            "MAX"
          END
            derV 0     @in case higher der. > 0
            ≤ NOT
          THEN
            "MIN"      @return "MIN"
          END
        END
        sol DTAG       @remove label "HIGHER"
        SWAP →TAG      @add label "MAX" or "MIN"
      END
    ELSE               @else (2nd. der. wasn't 0)
      sol              @simply return solution
    END
  >>
>>
DOSUBS
1                      @Do for all sub lists
<<
  f OVER SUBST         @find f(x) at extremum
  EXPAND "F(X)"        @Label and wrap in list
  →TAG 2 →LIST
>>
            DOSUBS
          END
        END
      >>
    >>
```

This is the (preliminary) code of FINDEX, a program for finding extrema of monovariate functions. (The program FINDEX that comes with this document has additional code, but we will examine the final version later.) The program takes the function from stack level 2 and its variable from stack level 1, and returns a list of extrema (if possible). Let's try it (as always). Enter $\mathrm{COSH}\left(X^2-1\right)$, then X, and press FINDEX. The HP49G works for some seconds and then it returns the list:

$$\{\{\mathrm{MIN}:(X=-1)\quad F(X)=1\}\quad\{\mathrm{MIN}:(X=1)\quad F(X)=1\}\quad\{\mathrm{MAX}:(X=0)\quad F(X)=\mathrm{COSH}(1)\}\}$$



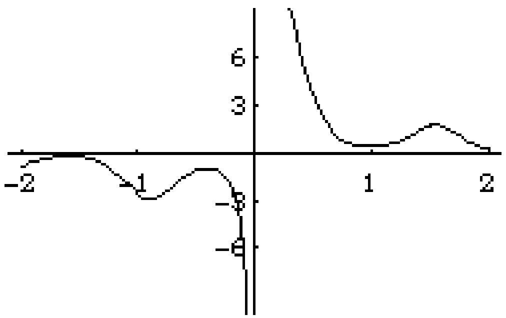
If you plot the function $\mathrm{COSH}\left(X^2-1\right)$ then you get something like the picture to the right an so you can see that the results are OK. The program can also handle monovariate functions that contain parameters *if you make assumptions for these parameters before you run it*. Let's try such a case. Enter the function $X^3+2\ a\ X^2-a^2\ X-2\ a^3$, and then enter the variable X. Before you run the program enter $a\geq 0$, and press ASSUME to specify that the parameter a is non negative. Drop the inequality from the stack, and press FINDEX. The HP49G takes the assumption for a into consideration and returns:

$$\left\{\begin{array}{l}\left\{\text{MAX}:\left(X=-\dfrac{\left(2+\sqrt{7}\right)\,a}{3}\right)\quad F(X):\dfrac{\left(-20+14\,\sqrt{7}\right)\,a^3}{27}\right\}\\ \left\{\text{MAX}:\left(X=\dfrac{\left(-2+\sqrt{7}\right)\,a}{3}\right)\quad F(X):-\dfrac{\left(20+14\,\sqrt{7}\right)\,a^3}{27}\right\}\end{array}\right\}$$

TABVAR would error out with this function, saying Parameters not allowed. The program FINDEX will also handle functions whose first and second derivatives are equal to 0, and which have some higher derivative of even order that is not equal to 0. Enter $a\ X^4$, then X, and then press FINDEX. The HP49G needs some seconds to return $\{\{\text{MIN}:(X=0)\quad F(X):0\}\}$. The program (in this preliminary version) will not handle functions whose first and second derivatives are equal to 0, and which have some higher derivative of odd order that is not equal to 0. But we add code for this purpose later on. Enter a now, and press UNASSUME to remove assumptions about variable a, and drop variable a from the stack.

If you are interested for a particular extremum rather than the global behaviour of the function, then you can use some of the built-in numeric solvers. For example, consider the function:

$$\frac{e^{\text{SIN}(5\ X)}}{X}$$



Its plot shows that it has a maximum somewhere around $X=1.5$. let's use the numeric function solver to find the maximum. Enter:

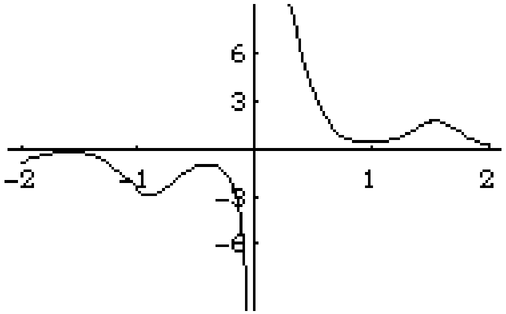
$$\frac{e^{\text{SIN}(5\ X)}}{X}$$

We must find a root of the first derivative somewhere around $X=1.5$, so we must find the first derivative first. Enter X and press ∂ and EXPAND to get:

$$\frac{\left(5\ X\ \text{COS}(5\ X)-1\right)\ e^{\text{SIN}(5\ X)}}{X^2}$$

Press APPS to get the pop up menu of built-in applications. Select 4. Numeric solver and press ENTER to get a new pop up menu with all built-in numeric solvers. The first menu item, 1. Solve equation is already selected, so press ENTER again to go to the SOLVE EQUATION screen. The input field Eq: is already selected and we must input there the first derivative of our function, which we left on the stack. Press HIST to go to the interactive stack, and then press ECHO. Press CANCEL to return to the SOLVE EQUATION screen. The derivative was put in the command line of the screen when you pressed ECHO. Press ENTER to put it in the input field Eq:. The HP49G selects automatically the next input field, which is X:. Enter the guess value 1.5 here, since the plot has shown us that the maximum occurs somewhere around $X=1.5$. The HP49G selects automatically the next input field, which is again Eq:. Press ▼ to move the selection to the input field X:. Now press SOLVE to solve (numerically) for X. After some seconds the HP49G returns 1.54483063848 in the input field X:.

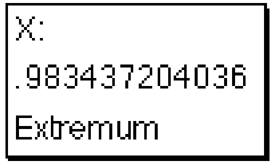
This is a root of the first derivative and an extremum (maximum) of the function. If you press INFO the HP49G displays a message box with information about the solution. As you can see, the HP49G didn't find a value for X for which the first derivative is 0., but it found a value which makes the first derivative almost equal to 0.. That means that it found two subsequent values for X, which differ 1E −12 from each other and for which the value of the first derivative changes its sign. Press OK to let the message box go away and return to the solver. Press ▲ to go to the input field Eq: and press EXPR=. The HP49G returns the value of the expression for the current value of X. Press CANCEL to return to the stack. Stack level 1 contains now Expr:(−2.59779932048E −11), which shows that we (presumably) have to do with root of the first derivative, i.e. with an extremum of the original function. On stack level 2 we have the solution labeled with X. This was returned to the stack when you pressed SOLVE.



In the above example we worked with the first derivative of the function. When we have a positive minimum or a negative maximum we can work with the function itself in the numeric solver. For example, the function $\frac{e^{SIN(5\,X)}}{X}$ has a positive minimum somewhere around $X = 1$, as the plot shows. Go to the SOLVE EQUATION screen again. The input field Eq: is selected and it contains the first derivative of our previous example. Press EQW and type:

$$\frac{e^{SIN(5\,X)}}{X}$$

Press ENTER to put the function in the input field Eq:. The HP49G moves the selection to X:. Enter 1. Press ▼ and then SOLVE. After some seconds the HP49G returns .983437204036 in the input field X:, and X:.983437204036 to the stack. Press INFO. The HP49G displays a message box again, which shows that an extremum was found. If you press ▲ to go to the input field Eq: and press EXPR= the HP49G returns the value of the expression for the current value of X. Press CANCEL to return to the stack. Stack level 1 contains now Expr:.381974744011, which shows that we have to do with a positive minimum of the function. Don't forget that you can find roots and extrema in the plotting environment too, as shown on page 1-2 of this marathon. You can also use the command ROOT to find extrema programmatically. Let's have an example. Press EQ to recall the current equation. Enter 'X', the variable to solve for, in single quotes. We use quotes because when we solved for X in the previous examples, the solution was stored in variable X. This is the behaviour of the numeric function solver, it always stores the found solution in the variable that we solved for. Enter 1, our guess value. Press ROOT and wait until the HP49G returns .983437204036. The command ROOT doesn't return any information about the solution. But of course we can substitute the solution in the function and see if it is 0. or almost 0., positive or negative. Purge variable X now so that it doesn't interfere with our work later on.



Let's continue now to another characteristic point of a function, the point of inflection. Using the HP49G we can find (one way or another) if some given function has an inflection point, and also where this point is. A function $f(x)$ has an inflection point at $x = \xi$ when:

$$\left.\frac{\partial^2 f}{\partial x^2}\right|_{x=\xi} = 0 \text{ and } \left.\frac{\partial^n f}{\partial x^n}\right|_{x=\xi} \neq 0 \text{ and } n > 2 \text{ and } n \text{ is odd.}$$

Let's see how what we can do to find inflection points. First of all, we can find the second derivative, then find its roots (if possible), and then examine if there is some higher derivative of odd order which is not equal to 0 at the roots of the second derivative. For example consider the function $\mathrm{TAN}(X)\cdot e^{-2X}$. Enter $\mathrm{TAN}(X)\cdot e^{-2X}$, then X, and then press [∂] to get the first derivative. Enter X and press [∂] again to get the second derivative. Expand it to get:

$$\frac{2\cdot\mathrm{TAN}(X)^3-4\cdot\mathrm{TAN}(X)^2+6\cdot\mathrm{TAN}(X)-4}{e^{2X}}$$

Press [ENTER] to make a copy of the expression on stack level 2. Now enter X and press [SOLVE] to find the roots of the second derivative. The HP49G returns:

$$X=\frac{4\cdot\pi\cdot n1+\pi}{4}$$

Now we will check to see if the third derivative is not equal to 0 at the above roots. Swap stack levels 1 and 2, enter X and press [∂] again, to get the third derivative. Expand it to get:

$$\frac{6\cdot\mathrm{TAN}(X)^4-12\cdot\mathrm{TAN}(X)^3+20\cdot\mathrm{TAN}(X)^2-20\cdot\mathrm{TAN}(X)+14}{e^{2X}}$$

Press [▲] once to go to the interactive stack, and another time to go to stack level 2. Press [ECHO] to copy the solution to the command line, [CANCEL] to leave the interactive stack, and [ENTER] to put the solution on stack level 1. Press [SUBST] to substitute $\frac{4\cdot\pi\cdot n1+\pi}{4}$ for X in the third derivative. Press [EXPAND] to convert all occurrences of:

$$\mathrm{TAN}\left(\frac{4\cdot\pi\cdot n1+\pi}{4}\right)$$

to:

$$\frac{\frac{\mathrm{SIN}(n1\cdot\pi)}{\mathrm{COS}(n1\cdot\pi)}+1}{1-\frac{\mathrm{SIN}(n1\cdot\pi)}{\mathrm{COS}(n1\cdot\pi)}}$$

The same old story again, we don't have INTEGERASSUME and so we must enter $\{\mathrm{SIN}(n1\cdot\pi)\ \ 0\}$ and press [↑MATCH], to convert all $\mathrm{SIN}(n1\cdot\pi)$ to 0 (since $n1$ is integer). Drop the $1.$ from the stack and press [EXPAND] to get:

$$\frac{8}{\left(e^{\frac{(4\cdot n1+1)\cdot\pi}{4}}\right)^2}$$

which is not equal to 0 for any integer value of $n1$. That means that the solutions $X=\frac{4\cdot\pi\cdot n1+\pi}{4}$ are indeed points of inflection of the function $\mathrm{TAN}(X)\cdot e^{-2X}$.

The above example was relatively easy, since the HP49G could solve the equation:

$$\frac{2\cdot\mathrm{TAN}(X)^3-4\cdot\mathrm{TAN}(X)^2+6\cdot\mathrm{TAN}(X)-4}{e^{2X}}=0$$

i.e. $\frac{\partial^2 f}{\partial x^2}=0$ without help. But there will be more than enough cases where this isn't possible. In these cases we can again try many different methods. Consider for example the function $\frac{\mathrm{SIN}(X)}{X}$ from the previous pages. If you differentiate it twice for X and expand, then you get:

$$\frac{(X^2-2)\ SIN(X)+2\ X\ COS(X)}{X^3}$$

The HP49G can't find the roots of this function analytically, so we can't find the points of inflection analytically. But if you plot it you can find out with the help of the root finder in the plotting environment that it has roots at approximately $n\pi$, where $n = 1,2,3,\ldots$. The smaller roots have the greatest deviation from $n\pi$, but as they go greater and greater they agree better and better with $n\pi$. Here are the first four positive roots for a demonstration of this fact.

Root	$n\pi$	
2,08157587782	3.14	$(n=1)$
5.94036999057	6.28	$(n=2)$
9.20584014294	9.42477796077	$(n=3)$
12.4044450219	12.5663706144	$(n=4)$

We already see that if we find some analytic approximation $r(n)$ of this behaviour, it has to approach $n\pi$ for greater values of n. Let's try to expand the second derivative

$$\frac{(X^2-2)\ SIN(X)+2\ X\ COS(X)}{X^3}$$

to a series at $n\pi$. If you use SERIES to expand the second derivative to a series, then it will presumably take a very long time, until the HP49G comes up with an answer. I tried to do that and had to interrupt the calculation after about half an hour (!). So we have to proceed differently. We can expand the function itself to a series of 4th order around $n\pi$, and differentiate the series twice to get a polynomial of second order. We want a series of second order because we know that the HP49G can solve analytically a polynomial of second order. (And also because we hope that it will be an adequate description of the second derivative in the neighbourhood of $n\pi$ - any science based on mathematics is a science based on hope. ;-)) Enter $\frac{SIN(X)}{X}$, then $X = n\pi$, and then 4 (the order), and press SERIES.

After 2.7 minutes (!) the HP49G returns a list in stack level 2 and the equation $h = X - n\pi$ on stack level 1. Press SUBST to substitute $X - n\pi$ for h in all expressions contained in the list. We need only the series expansion, which is the third element in the list, so enter 3 and press GET to extract it from the list. Press DTAG to get rid of the label. Now you have:

$$-\frac{(n^4\pi^4-12\ n^2\pi^2+24)\ SIN(n\pi)+(4\ n^3\pi^3-24\ n\pi)\ COS(n\pi)}{24\ n^6\pi^6}(X-n\pi)^5$$
$$+\frac{(n^4\pi^4-12\ n^2\pi^2+24)\ SIN(n\pi)+(4\ n^3\pi^3-24\ n\pi)\ COS(n\pi)}{24\ n^5\pi^5}(X-n\pi)^4$$
$$+\frac{(3\ n^2\pi^2-6)\ SIN(n\pi)-(n^3\pi^3-6\ n\pi)\ COS(n\pi)}{6\ n^4\pi^4}(X-n\pi)^3$$
$$-\frac{(n^2\pi^2-2)\ SIN(n\pi)+2\ n\pi\ COS(n\pi)}{2\ n^3\pi^3}(X-n\pi)^2$$
$$-\frac{SIN(n\pi)-n\pi\ COS(n\pi)}{n^2\pi^2}(X-n\pi)$$
$$+\frac{SIN(n\pi)}{n\pi}$$

The series is of 5th order, so press ▼ to get the expression in the EQW, press ▼ again to select the first term, and then press DEL to delete the 5th order term. (The 4th order term will be automatically selected after deletion of the 5th order term.) Press ENTER to put the series back to the stack. For $n = 1,2,3\ldots$ the expression $SIN(n\pi)$ is equal to 0. Enter the list $\{SIN(n\pi)\ 0\}$ and press ↑MATCH. Drop the 1. and expand. Enter X and press REORDER to sort for powers of X. Now stack level 1 contains:

$$\frac{\begin{pmatrix}(\mathrm{COS}(n\cdot\pi)\cdot\pi^2\cdot n^2-6\cdot\mathrm{COS}(n\cdot\pi))\cdot X^4\\ -(5\cdot\mathrm{COS}(n\cdot\pi)\cdot\pi^3\cdot n^3-30\cdot\mathrm{COS}(n\cdot\pi)\cdot\pi\cdot n)\cdot X^3\\ +(9\cdot\mathrm{COS}(n\cdot\pi)\cdot\pi^4\cdot n^4-60\cdot\mathrm{COS}(n\cdot\pi)\cdot\pi^2\cdot n^2)\cdot X^2\\ -(7\cdot\mathrm{COS}(n\cdot\pi)\cdot\pi^5\cdot n^5-60\cdot\mathrm{COS}(n\cdot\pi)\cdot\pi^3\cdot n^3)\cdot X\\ +2\cdot\mathrm{COS}(n\cdot\pi)\cdot\pi^6\cdot n^6-24\cdot\mathrm{COS}(n\cdot\pi)\cdot\pi^4\cdot n^4\end{pmatrix}}{6\cdot\pi^4\cdot n^4}$$

Now we will find the second derivative of this expression. Enter X, press ∂, enter X, and press ∂ . Now expand, enter X and press REORDER to get:

$$\frac{\begin{pmatrix}(2\cdot\mathrm{COS}(n\cdot\pi)\cdot\pi^2\cdot n^2-12\cdot\mathrm{COS}(n\cdot\pi))\cdot X^2\\ -(5\cdot\mathrm{COS}(n\cdot\pi)\cdot\pi^3\cdot n^3-30\cdot\mathrm{COS}(n\cdot\pi)\cdot\pi\cdot n)\cdot X\\ +3\cdot\mathrm{COS}(n\cdot\pi)\cdot\pi^4\cdot n^4-20\cdot\mathrm{COS}(n\cdot\pi)\cdot\pi^2\cdot n^2\end{pmatrix}}{\pi^4\cdot n^4}$$

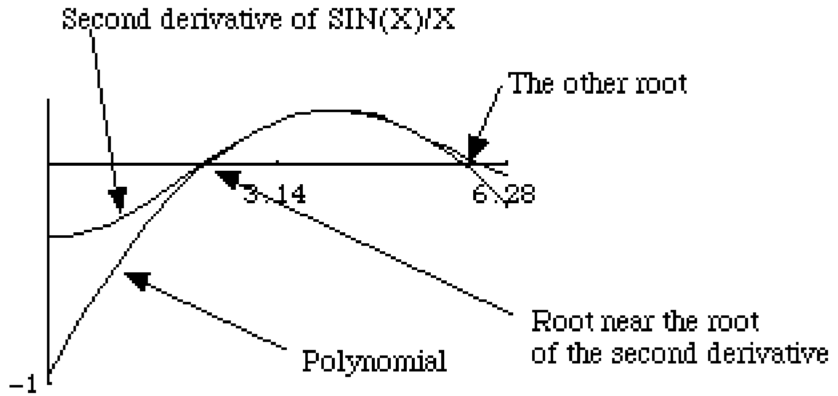
This is a polynomial of degree 2, for which the HP49G can find analytical solutions. Before we find the solutions, we simplify it a bit more. All terms have a common factor of $\mathrm{COS}(n\cdot\pi)$. The expression $\mathrm{COS}(n\cdot\pi)$ can be 1 or -1. No matter which of both we choose, the above polynomial will have the same roots, because we just multiply it with 1 or -1. As long as we are only interested for the roots, it doesn't matter if we replace $\mathrm{COS}(n\cdot\pi)$ with 1, or with -1. So we choose to replace $\mathrm{COS}(n\cdot\pi)$ with 1. Enter the list $\{\mathrm{COS}(n\cdot\pi)\ 1\}$ and press ↑MATCH. Drop the $1.$ from the stack, expand and reorder for X. Now the polynomial is:

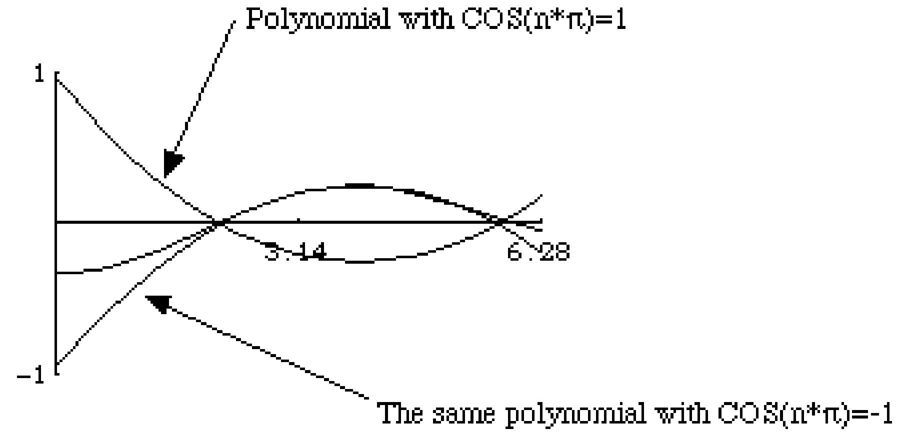
$$\frac{(2\cdot\pi^2\cdot n^2-12)\cdot X^2-(5\cdot\pi^3\cdot n^3-30\cdot\pi\cdot n)\cdot X+3\cdot\pi^4\cdot n^4-20\cdot\pi^2\cdot n^2}{\pi^4\cdot n^4}$$

The HP49G can find the roots of this polynomial for X. Enter X and press SOLVE. After some seconds you get the list:

$$\left\{\begin{array}{l}X=\dfrac{5\cdot n^3\cdot\pi^3-30\cdot n\cdot\pi-n\cdot\pi\cdot\sqrt{n^4\cdot\pi^4+4\cdot n^2\cdot\pi^2-60}}{4\cdot n^2\cdot\pi^2-24}\\ X=\dfrac{5\cdot n^3\cdot\pi^3-30\cdot n\cdot\pi+n\cdot\pi\cdot\sqrt{n^4\cdot\pi^4+4\cdot n^2\cdot\pi^2-60}}{4\cdot n^2\cdot\pi^2-24}\end{array}\right\}$$

Of these two roots it is the first that we need. The other one is the second root of the quadratic polynomial, as the picture below illustrates. Actually the polynomial which we obtained replacing $\mathrm{COS}(n\cdot\pi)$ with 1 is exactly the negative of the parabola shown in the picture below. But it still has the same roots, which is the only thing that we want in this example. Take a look at the picture on the next page to understand better how the plots of the polynomials with $\mathrm{COS}(n\cdot\pi)=1$ and with $\mathrm{COS}(n\cdot\pi)=-1$ relate to each other. Let's see how well the found solution represent the roots of the second derivative of $\dfrac{\mathrm{SIN}(X)}{X}$, i.e.





how well they represent the inflection points of $\frac{\mathrm{SIN}(X)}{X}$. Press HEAD to extract the first element of the list, press EQ→ to separate the right form the left hand side of the equation, press ▶ to swap stack levels 1 and 2, and then press ◀ to drop the X from the stack. Press ENTER to make a copy of the expression. Let's make a sequence of the expression for $n = 1$ to $n = 4$. Enter n, 1, 4, 1, and press SEQ. After some seconds the HP49G returns a list that contains the results of the expression:

$$\frac{5\ n^3\ \pi^3 - 30\ n\ \pi - n\ \pi\sqrt{n^4\ \pi^4 + 4\ n^2\ \pi^2 - 60}}{4\ n^2\ \pi^2 - 24}$$

for $n = 1,2,3,4$. Now we will convert all expressions in the list to numbers. Enter the program `<< →NUM >>` and press MAP. The HP49G returns a list that contains 4 numbers, which are very good approximations of the roots of the second derivative of $\frac{\mathrm{SIN}(X)}{X}$.

Root	Approximated root
2,08157587782	2.14727275738
5.94036999057	5.9443688354
9.20584014294	9.20724109123
12.4044450219	12.4050705181

Drop the list from the stack and let's look again at the analytic approximation of the roots of the second derivative, i.e. the analytic approximation of the inflection points of the function $\frac{\mathrm{SIN}(X)}{X}$. When n goes to greater values, the expression:

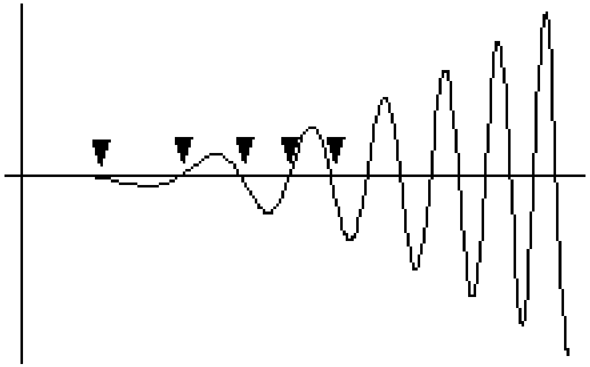
$$\frac{5\ n^3\ \pi^3 - 30\ n\ \pi - n\ \pi\sqrt{n^4\ \pi^4 + 4\ n^2\ \pi^2 - 60}}{4\ n^2\ \pi^2 - 24}$$

goes to:

$$\frac{5\ n^3\ \pi^3 - n\ \pi\sqrt{n^4\ \pi^4}}{4\ n^2\ \pi^2} = n\ \pi$$

Our analytic approximation does indeed a good job.

Let's do another example that the HP49G can't solve out of the box, but which shows how important it can be, to "smell" mathematics. We want to find an analytic approximation of the inflection points of $\mathrm{SIN}(X^2)$. enter $\mathrm{SIN}(X^2)$, and differentiate twice for X. Expand to get $-(4\ X^2\ \mathrm{SIN}(X^2) - 2\ \mathrm{COS}(X^2))$. Store that in some variable as we are going to use it again later. The HP49G can't find the roots of this expression analytically. (Again, who can?) If you plot this with horizontal view range from $0.$ to 6.28 and vertical view range from $-150.$ to $150.$, then you get a plot that looks like the picture on the next page. As you can see the roots are always denser as X grows. While in the plotting environment, press FCN, move the graphics cursor near the location indicated by the first arrow from the left, and press ROOT. After some seconds the HP49G displays the root and puts a copy of it on stack level 1. Move the cursor to the second arrow, press some menu key to display the menu again, and press ROOT again. Repeat



this for the remaining three roots. Press CANCEL twice to return to the stack. Now you have the five roots on the stack. Press ▲ to go to the interactive stack, then press ▲ four times to go to stack level 5, press NXT to get the second row of the menu of the interactive stack, and then press →LIST to put all roots in one list. Press CANCEL to return to the stack. We will use Newton' method for finding analytic approximations of the roots, so we need good "guess" values for the method. How do they relate to π, these roots? Press ENTER to make a copy of the list and let's "smell" mathematics. Press x^2 to get the squares of all numbers in the list. Press π and then →NUM to get the numeric approximation of π. Press ÷ to divide all number in the list by the numeric approximation of π. If you now press ▼ to view the list, you will see that all numbers except the first are approximately integers. Which means that the squares of the roots are (almost) divisible by π. If we denote some root with r, then we have:

$$r^2 = n\ \pi \quad r = \pm\sqrt{n\ \pi}$$

Can you imagine how this could be "smelled"? Anyway, for us this means that we can use $\sqrt{n\ \pi}$ as a guess value for getting an analytic approximation of the roots of $-\left(4\ X^2\ \mathrm{SIN}\left(X^2\right)-2\ \mathrm{COS}\left(X^2\right)\right)$, i.e. the inflection points of $\mathrm{SIN}\left(X^2\right)$. Drop the list form stack level 1. We are going to construct $X_1 = X_0 - \dfrac{F(X_0)}{F'(X_0)}$ with $F(X) = -\left(4\ X^2\ \mathrm{SIN}\left(X^2\right)-2\ \mathrm{COS}\left(X^2\right)\right)$. Enter $\sqrt{n\ \pi}$. Then recall the expression $-\left(4\ X^2\ \mathrm{SIN}\left(X^2\right)-2\ \mathrm{COS}\left(X^2\right)\right)$, enter $X = \sqrt{n\ \pi}$ and press SUBST. Expand the result. Enter $\{\mathrm{SIN}(n\ \pi)\ 0\}$ and press ↑MATCH. Drop the 1. from the stack and expand. Stack level 1 now contains $2\ \mathrm{COS}(n\ \pi)$, which is $F(X_0)$. Recall the expression $-\left(4\ X^2\ \mathrm{SIN}\left(X^2\right)-2\ \mathrm{COS}\left(X^2\right)\right)$ and take its derivative for X. Enter $X = \sqrt{n\ \pi}$ and press SUBST and expand. Enter again $\{\mathrm{SIN}(n\ \pi)\ 0\}$ press ↑MATCH, and drop the 1. from the stack. Expand to get $-\left(8\ n\ \pi\sqrt{n\ \pi}\ \mathrm{COS}(n\ \pi)\right)$. This is $F'(X_0)$. Press ÷ and expand. Press −. Now you have the expression:

$$\sqrt{n\ \pi}+\frac{\sqrt{n\ \pi}}{4\ n^2\ \pi^2}$$

This is our analytic approximation of the roots of $-\left(4\ X^2\ \mathrm{SIN}\left(X^2\right)-2\ \mathrm{COS}\left(X^2\right)\right)$, i.e. the inflection points of $\mathrm{SIN}\left(X^2\right)$. Let's try it for some values of n. Press ENTER to make a copy of it, enter n, 1, 4, 1, and press SEQ. Enter the program `<< →NUM >>` and press MAP. You get a list with numbers that are very close to those in the list of roots that we found in the plotting environment. Only the first root that we found there, namely .808251932936, is not in the list of numbers that result from the analytic approximation. Can you imagine why? (What is the slope of the function

$-\left(4\ X^2\ \mathrm{SIN}\left(X^2\right)-2\ \mathrm{COS}\left(X^2\right)\right)$ at $X=\sqrt{0\ \pi}$, i.e. $X=0$? - read Aaron's posting again.)

Before going further, there are a couple of things here, that are worth saying. The above examples of analytic approximations may give the impression that we did modelling. The truth is that we didn't, if with "modelling" we mean physical modelling. We simply took advantage of known facts and used them to replace something that we can't find exactly with an approximation of that something. The approximation might have an analytic closed form, but this doesn't imply that we did physical modelling. Consider for example the function $\frac{\mathrm{SIN}(X)}{X}$ that we used in some examples. Suppose it describes mathematically some physical quantity that occurs in some physical phenomenon. We have found (approximately) that the second derivative (i.e. curvature) of this physical quantity will be equal to 0 at $\sqrt{n\ \pi}+\frac{\sqrt{n\ \pi}}{4\ n^2\ \pi^2}$ where $n=1,2,3$, i.e. we introduced a new variable n that represents (is?) some other quantity, which can only have integer values greater than 0. This result is only the consequence of a model which we didn't even think about. The process of modelling precedes even the usage of the function $\frac{\mathrm{SIN}(X)}{X}$. For example, having a spring and assuming that the force which acts on a mass connected to the end of the spring is proportional to the spring's amplitude (i.e. distance of the springs end from the equilibrium point) is modelling. Obtaining some function (or any other mathematical object) that describes the phenomenon is rather following the consequences of the model, than modelling itself. We make (a minimum of) assumptions and follow their consequences the mathematical way, in order to achieve a maximum of details of the description of the phenomenon. These details are theoretical predictions, which have to be proved by experiments. If we don't do them, the theory is unproved and nobody in this world (including JHM ;-)) can say that the theory is the absolute truth about the world. If we do them and their results contradict what we predicted theoretically, then the theory is false! If we do them and their results agree with our predictions, then the theory is… usable! It is still not the truth! Why? Because modelling, reasonable modelling, abstracts from the real existing matter, and creates such ideal meanings, which might not even exist. To stay in the example with the spring, Hook's law takes birth by using only two quantities, the length of the feather, and its "stiffness", ignoring any other property that the spring might have. Modelling contains (always?) this simplification. When we do experiments, we prove the theoretically predicted quantities that arise from the (simplified) model. Now, changes are that we will not measure exactly what the theory predicts. There will be deviations. And the question is, are these deviations the result of the simplification, of the abstraction, or do they have other reasons? (Like for example Nick's catastrophic hands in a laboratory ;-)) If we are able to exclude the "other reasons", i.e. Nick's hands, we still accept the model not because it lets us calculate physical quantities with infinite precision, but rather because a reasonable amount of precision *and* a model that is easy to understand, are together something that we can easily grasp, an understandable theory that gives us means to falsify it. Don't underestimate the value of falsification possibilities. It is exactly *this* that makes a theory (and the underlying model) a usable theory, which in some extend describes reality. Without falsification possibilities, like for example experiments, the model and the theory is… pa-par-la-pap! Words without any value. And even if the theory withstands all falsification experiments, nobody can assure us that there will not be somebody that finds an experiment that successfully falsifies the theory. This is one of the reasons why you won't here physicists (I mean physicists that deserve their name) talking about "the truth" about the world. This is something that they leave for JHM. ;-) End of philosophy, back to calculus.

Unfortunately the command TABVAR doesn't include any inflection points of a function. It leaves them out. So we can write a program that finds them. But before we do that, we make a minor correction in the program FINDEX. The program will crash, if at some point the second derivative of the function is 0, and at the same time the next derivative that is not equal to 0 is of odd order. So we add code that takes care of this case. Turn page for the corrected listing which includes the additional code in red colour.

```
…………………
IF
  n 2 MOD NOT @If der. order is even
THEN
  CASE        @in case higher der. < 0
      derV 0
      ≥ NOT
    THEN      @return "MAX"
      "MAX"
    END
      derV 0  @in case higher der. > 0
      ≤ NOT
    THEN
      "MIN"   @return "MIN"
    END
  END
  sol DTAG    @remove label "HIGHER"
  SWAP →TAG   @add label "MAX" or "MIN"
ELSE          @else (der. order is odd)
  {}
END
…………………
```

```
…………
1                       @Do for all sub lists
<<
  IF
    DUP {} ≠
  THEN
    f OVER SUBST EXPAND @find f(x) at extremum
    "F(X)" →TAG 2 →LIST @Label and wrap in list
  END
>>
DOSUBS
……………
```

This version, which is also the version that comes with this document, will return an empty list for each value of the function variable, where the second derivative vanishes, and where the first derivative that doesn't vanish is of odd order. Not an elegant method, but at least it takes care of such cases. Now on to the program FINDINFL, which tries to find inflection points. It is quite similar to FINDEX. It tries to find the roots of the second derivative of the function, and if it succeeds, it checks the higher derivatives until it finds one that is not equal to 0 at the root(s). If the found derivative is of odd order, the program returns the found inflection point.

```
<<
  NOVAL {}
  → f v f´´ sols
  <<
    f v ∂ v ∂ EXPAND
    'f´´' STO               @Find 2nd. der., store in f´´
    f´´ v                   @Prepare for SOLVE
    IFERR                   @If SOLVE errors out
      SOLVE
    THEN                    @then clean up and return
      DROP2 f v             @user input
      "FINDINFL Error:      @and mimic system errors
Can't solve f´´=0
Reason:
"
      ERRM + 1 DISP
      1200 .08 BEEP
      3 FREEZE
    ELSE                    @else (SOLVE worked)
      IF                    @If solution was empty list
        DUP {} SAME
      THEN                  @then clean up and return
        DROP f v            @user input
        "FINDINFL Error:
No solutions of f´´=0
were found"
        1 DISP 1200 .08 BEEP
        3 FREEZE              @Mimic system errors
      ELSE                    @else (solutions weren't {})
        IF                    @If solutions weren't a list
          DUP TYPE 5 ≠
        THEN                  @then convert them to a list
          1 →LIST
        END
```

```
        'sols' STO
        sols
        1                    @Do for all solutions
        <<
          NOVAL 2 f´´
          →  sol derV n derF
          <<
            WHILE            @Keep on finding higher
              derF v ∂       @higher derivatives and
              'derF' STO     @substituting the solution
              derF sol       @and updating the derivat.
              SUBST          @order
              'derV' STO     @until you find derivative
              1 'n' STO+     @that is different from 0
              derV 0 ==
            REPEAT
            END
            IF
              n 2 MOD        @If der. order is odd
            THEN             @then label solution,
              sol "INFL"     @find f(x) and label it,
              →TAG           @and wrap both in a list.
              f sol SUBST
              EXPAND
              "F(X)" →TAG
              2 →LIST
            ELSE             @else (der. order is even)
              {}             @return empty list
            END
          >>
        >>
        DOSUBS
      END
    END
  >>
>>
```

Let's try the program. Enter:

$$e^{-\frac{1}{X}}$$

then enter X, and then press FINDINFL. After a couple of seconds you get:

$$\left\{\left\{\text{INFL}: X=\frac{1}{2}\ \ F(X):\frac{1}{e^{2}}\right\}\right\}$$

Of course the two programs FINDEX and FINDINFL will gasp a lot when they have to do with solutions that contain arbitrary integers, like those returned by SOLVE for trigonometric functions. The reason is (as always ;-)) that there is no INTEGERASSUME in the HP49G. We could use the implementation of INTEGERASSUME that was introduced on page 2-71 of the first volume of the Basic Calculus Marathon. But the problem will be that one has to check many patterns. For example suppose that you want to find the inflection points of $\text{SIN}(X)$ using the program FINDINFL. The program will find that the second derivative of $\text{SIN}(X)$ is $-\text{SIN}(X)$. Then it will solve $-\text{SIN}(X)$ for X, and so it will obtain:

$$\{X=-(2\cdot\pi\cdot n1-\pi)\quad X=2\cdot\pi\cdot n1\}$$

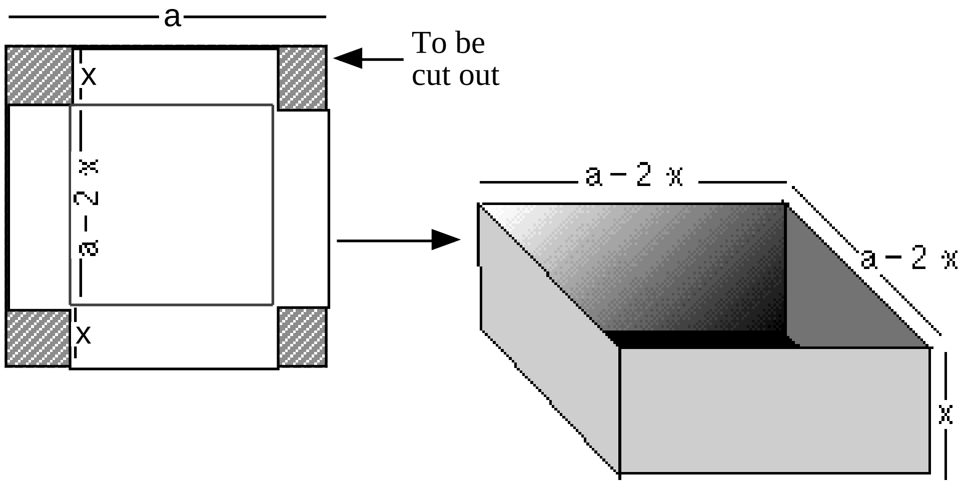
Now the program has to find out if the next derivative, $-\text{COS}(X)$, is (or isn't) equal to 0 at $X=-(2\cdot\pi\cdot n1-\pi)$ and $X=2\cdot\pi\cdot n1$. It will substitute these solutions in $-\text{COS}(X)$, obtaining $-\text{COS}(-(2\cdot\pi\cdot n1-\pi))$ and $-\text{COS}(2\cdot\pi\cdot n1)$. Consider the second of these formulae. We have to somehow teach the HP49G that in this case if $n1$ is integer, then $-\text{COS}(2\cdot\pi\cdot n1)$ will be equal to -1, no matter if $n1$ is odd or even. We could simply add $n1$ to INTEGERASSUME, and use the proper pattern matching list along with ISINTEG?, to match it to -1. But then there is one danger. If we try to find the inflection points of, say $\text{SIN}\left(\frac{2\cdot X}{3}\right)$, then one of the roots of the second derivative will be $X=3\cdot\pi\cdot n1$. If we then match $-\text{COS}(3\cdot\pi\cdot n1)$ to -1 the same way as above, then we will be making a mistake, since

$-COS(3\ \pi\ n1)$ is equal to 1 only for even values of $n1$. In this case it doesn't matter, since both 1 and -1 are different than 0, but for more complex functions it might be that we match something that is different from 0 to 0 or vice versa. So the problem still remains.

Let's have now some examples of the usage of derivatives in "real world", whatever "real" might mean. We will consider some examples, in which we can maximise or minimise one quantity that is a function of another quantity. We start with an easy example. We have a square piece of cardboard with the side length a. Cutting out four square pieces from the corners and folding the remaining piece we want to make a box. The box should have the biggest possible volume. How big must then x be? The volume of the box will be $(a-2\ x)^2\ x$. Since the volume of the box has to be a maximum, we must solve the equation:

$$\frac{\partial}{\partial x}\left((a-2\ x)^2\ x\right)=0$$

for x. Enter the above equation and take care to enter all x's small.



Enter x (also small) and press SOLVE. The HP49G returns:

$$\left\{x=\frac{a}{2}\quad x=\frac{a}{6}\right\}$$

It is easy to recognise that we need the second solution, $x=\frac{a}{6}$, since the first would simply cut the cardboard in four equal pieces, making the volume of the box to be 0. It is clear that finding the roots of the first derivative, we find extrema, minima or maxima of the function. If we want to mathematically prove if we have a minimum or a maximum, then we need the second derivative. Enter $(a-2\ x)^2\ x$, and differentiate twice for x. Expand to get $-(8\ a-24\ x)$. Press OVER to get a copy of the solutions list from stack level 2 to stack level 1. Press SUBST and then EXPAND to get $\{4\ a\quad -(4\ a)\}$. Since $a>0$, we have $4\ a>0$ and $-(4\ a)<0$, which means that the first solution, $x=\frac{a}{2}$, minimises the volume, and the second solution maximises the volume. Drop the list from stack level 1. How big will be the volume of the box when $x=\frac{a}{2}$ or $x=\frac{a}{6}$? Enter the expression $(a-2\ x)^2\ x$, press OVER, then SUBST, then expand to get:

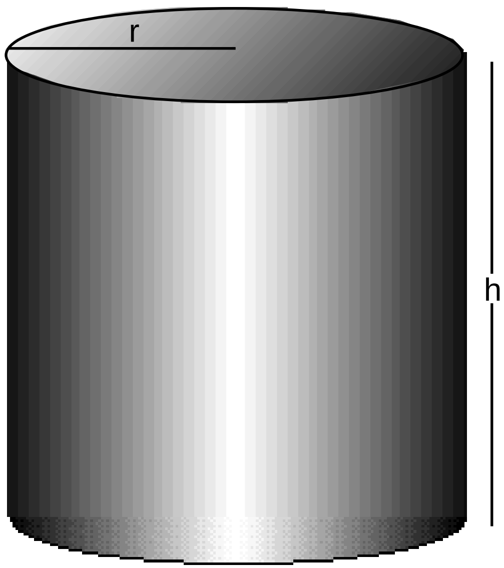
$$\left\{0\quad \frac{2\ a^3}{27}\right\}$$

which shows that indeed the first solution gives us a box of volume 0, i.e. no box at all. The second gives us a box with the maximum possible volume, $\frac{2\ a^3}{27}$. The same results we can get using the program FINDEX. Clear the stack first. Before we use the program, we have to make the right assumptions about the parameter a, which appears in the expression of the volume of the box. Enter $a \geq 0$, and press ASSUME. Drop the remaining expression $a \geq 0$ from the stack. Now enter $(a-2x)^2 x$, then enter x, and then press FINDEX. The HP49G needs 34 seconds to return:

$$\left\{\left\{MIN: \left(x=\frac{a}{2}\right) 0\right\} \left\{MAX: \left(x=\frac{a}{6}\right) \frac{2a^3}{27}\right\}\right\}$$

which also shows that the volume will be maximised, when x is one sixth of the side length of the cardboard. Now we can remove the assumptions for a. Enter a, press UNASSUME, and then press ◄ to drop the remaining a from the stack.

The second example belongs to the classics. I saw it for the first time in the manual of the legendary HP41. It was fun to read and since the sentimental remembrance (unfortunately) will not leave us oldies in peace, I will use it here. We want to make a cylindrical metal can with a certain volume V and use the minimum possible amount of metal.



(Though I would prefer the maximum amount of heavy metal, but that's another story ;-)) We use the minimum possible amount of metal sheet when we minimise the surface S of the cylinder which is given by:

$$S = 2\pi r^2 + 2\pi r h$$

Enter $2\pi r^2 + 2\pi r h$. Press ENTER to make a copy of the expression. Since the volume is given by:

$$V = \pi r^2 h$$

we can solve the last equation for h, in order to transform it to a function of r. Enter $V = \pi r^2 h$, press ENTER to make a copy of the equation, enter h and press SOLVE to get:

$$h = \frac{V}{r^2 \pi}$$

Press ▲ once to go to the interactive stack, and then again ▲ twice to go to stack level 3. Press ROLL to bring the expression $2\pi r^2 + 2\pi r h$ to stack level 1. Now press ▼ to go to stack level 2, and then press PICK to make a copy of the equation $h = \frac{V}{r^2 \pi}$ in stack level 1. Press CANCEL to return to the stack. Now press SUBST and then EXPAND to get:

$$\frac{2r^3\pi + 2V}{r}$$

This is the surface of the cylinder as a function of its radius r, and V being a parameter which has some arbitrary but constant value. (I.e. $1000cm^3$ or $345m^3$ or whatever.) Enter r, press ∂ and then EXPAND to get:

$$\frac{4r^3\pi - 2V}{r^2}$$

Press ENTER to make a copy of the expression. Enter again r, press ∂ and then EXPAND to get:

$$\frac{4r^3\pi + 4V}{r^3}$$

This is the second derivative which we will use when we want to tell maximum from minimum. Press ▶ to swap stack levels 1 and 2, enter r, and press SOLVE. The result is:

$$r = \sqrt[3]{\frac{V}{2\cdot\pi}}$$

Let's see if this is a minimum or a maximum. Press ▶ and then OVER. Now press SUBST to substitute the root of the first derivative in the second derivative and then EXPAND to get:

$$12\cdot\pi$$

Since $12\cdot\pi > 0$ we have a minimum, i.e. what we want. Drop $12\cdot\pi$ from the stack. Press ▶, then OVER. Press SUBST and then EXPAND to find the height of the cylindrical can:

$$h = \frac{V}{\pi\cdot\sqrt[3]{\frac{V}{2\cdot\pi}}^{2}}$$

What is the surface of the can? Press ▲ once to go to the interactive stack, and then again ▲ three times to go to stack level 4. Press ROLL to bring the expression $2\cdot\pi\cdot r^2+2\cdot\pi\cdot r\cdot h$ to stack level 1. Now press ▼ to go to stack level 3, and then press PICK to make a copy of the equation $r = \sqrt[3]{\frac{V}{2\cdot\pi}}$ in stack level 1. Press CANCEL to return to the stack. Now press SUBST to get:

$$2\cdot\pi\cdot\sqrt[3]{\frac{V}{2\cdot\pi}}^{2}+2\cdot\pi\cdot\sqrt[3]{\frac{V}{2\cdot\pi}}\cdot h$$

Now press ▲ to go to the interactive stack once again, and then again ▲ to go to stack level 2. Press PICK to make a copy of the equation $h = \frac{V}{\pi\cdot\sqrt[3]{\frac{V}{2\cdot\pi}}^{2}}$ in stack level 1. Press CANCEL to return to the stack.

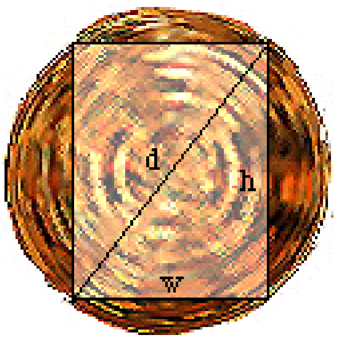
Now press SUBST and then EXPAND to get:

$$\frac{2\cdot\pi\cdot\sqrt[3]{\frac{V}{2\cdot\pi}}^{3}+2\cdot V}{\sqrt[3]{\frac{V}{2\cdot\pi}}}$$

This is not completely expanded, so press EXPAND again to get:

$$\frac{3\cdot V}{\sqrt[3]{\frac{V}{2\cdot\pi}}}$$

Clear the stack and let's go on to the next example. We want to cut a beam out of a trunk. The trunk's cross-section can be assumed to be circular with the diameter d. BTW, there is no single tree on this planet that has exactly circular cross-section but nonetheless we assume that. Modelling and simplification, you know! The beam will have the cross section of a rectangle with width w and height h. The ability T of the beam to carry weight is given by the equation:



$$T = c\cdot w\cdot h^2$$

where c is a constant, in which all material dependent properties hide. The question is: at which width and height does the beam have the maximum ability to carry weight? Before we solve this problem on the

HP49G, let's take a look at our model. As already said, no tree on this planet has a circular cross-section. Thus, it is not really exact to speak about the "diameter" of the cross-section. Nonetheless the model, and the results of the model, are usable. We abstract from the tree trunk a property which… almost exist. We idealise the trunk and make a cylinder out of it. *That means that the results that we will get if we follow the mathematics of the model have a certain grade of similarity with reality, but they are not reality*. They are a usable approximation of reality. And many times (if not always) it is exactly this process of idealisation that enables us to somehow understand more about the world. If we take this strict criticism to the limit, we can see that even something as simple as an integer, can be considered to be an idealisation, something that takes birth in our minds before we project it onto the "world". Think about it. It exists in our minds - the idea, the concept - but does it really exist "out there"? Nonetheless, even if we could say with absolute certainty that it doesn't really exist, it *is* the idealisation that creates self-contained stable models, which are usable. (Or try to determine the width and the height of a beam with maximum ability to carry weight, if you take the real cross-section of the trunk, which might be just about anything.)

After the philosophy there comes mathematics. Enter $c\ w\ h^2$. This quantity depends on two variables (c is a constant). We want to convert it to a function of a single variable. And that is where a constraint comes. No matter how long w and h are, the relation $d^2 = w^2 + h^2$ is always true. We can use it to find h as a function of w and substitute it in $c\ w\ h^2$. Now, we know that all variables and constants in the above expressions are real and greater than 0. Let's tell that the HP49G. Enter $\{c \geq 0 \quad d \geq 0 \quad w \geq 0 \quad h \geq 0\}$ and press **ASSUME**. Drop the list from stack level 1. Enter $d^2 = w^2 + h^2$, then h and press **SOLVE**. Though we explicitly told the HP49G that h is greater than 0, it returns the solution list:

$$\left\{h = -\sqrt{d^2 - w^2} \quad h = \sqrt{d^2 - w^2}\right\}$$

We only need the second solution. Enter 2 and press **GET** to extract it from the list. Press **DUP2** to make copies of the objects on stack levels 1 and 2. Now press **SUBST** and **EXPAND**. The result is $\left(w\ d^2 - w^3\right)\ c$.

Let's use FINDEX in this example. The function $\left(w\ d^2 - w^3\right)\ c$ is a monovariate function with the additional parameters d and c, for which we have made assumptions. Enter w and press **FINDEX**. The HP49G needs some seconds to return:

$$\left\{\begin{array}{l}\left\{\text{MIN:}\left(w = -\left(\frac{d}{3}\ \sqrt{3}\right)\right) \quad \text{F(X):}\left(-\frac{2\ \sqrt{3}\ d^3\ c}{9}\right)\right\} \\ \left\{\text{MAX:}\left(w = \frac{d}{3}\ \sqrt{3}\right) \quad \text{F(X):}\frac{2\ \sqrt{3}\ d^3\ c}{9}\right\}\end{array}\right\}$$

As we see the maximum ability to carry weight is found for $w = \frac{d}{3}\ \sqrt{3}$ and it is given by:

$$\frac{2\ \sqrt{3}\ d^3\ c}{9}$$

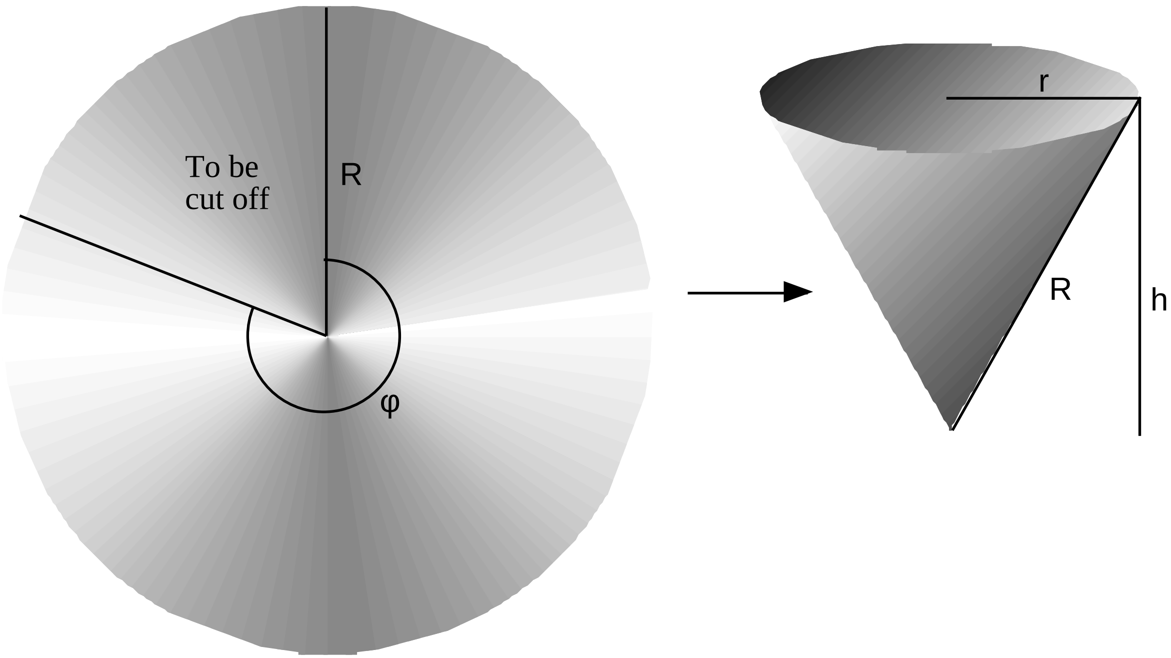
Enter 2 and press **GET** to extract the second sub list. Now we need to substitute $w = \frac{d}{3}\ \sqrt{3}$ in $h = \sqrt{d^2 - w^2}$ in order to find the height of the beam. Press **ENTER** to make a copy of the list, then **HEAD** to extract the object $\text{MAX:}\left(w = \frac{d}{3}\ \sqrt{3}\right)$ out of the list. Press **ROT** to bring the equation $h = \sqrt{d^2 - w^2}$ to stack level 1, and then ▶ to swap stack levels 1 and 2. Now press **SUBST** to make the substitution and then **EXPAND** to get:

$$h = \frac{\sqrt{6}\ d}{3}$$

Now we can remove the assumptions that we made. Enter $\{c\ \ d\ \ w\ \ h\}$, press UNASSUME, and drop the list from stack level 1.

We continue with another example. Through cutting a sector off from a metal disc with radius R, and through wrapping the rest conically, we want to make a funnel. The funnel should have the greatest possible capacity. How big must then the piece be that we cut off? We idealise the funnel to a cone. The capacity, i.e. volume of the funnel is given by:

$$\frac{\pi}{3}\ r^2\ h$$



where r is the radius of the funnel and h is its height. The radius r of the funnel and the radius R of the metal disc are connected to each other by the relation:

$$r = \sqrt{R^2 - h^2}$$

Enter the volume of the funnel:

$$\frac{\pi}{3}\ r^2\ h$$

then enter $r = \sqrt{R^2 - h^2}$ and press SUBST and EXPAND to get:

$$\frac{\left(h\ R^2 - h^3\right)\ \pi}{3}$$

The radius of the disc, R, is positive. Enter $R \geq 0$ and press ASSUME. Drop the inequality form the stack and enter h (the variable). Press FINDEX to get:

$$\left\{ \begin{array}{l} \left\{ \text{MIN:}\left(h = -\left(\frac{R}{3}\ \sqrt{3}\right)\right)\ \ \text{F(X):}\left(-\frac{2\ \sqrt{3}\ R^3\ \pi}{27}\right) \right\} \\ \left\{ \text{MIN:}\left(h = \frac{R}{3}\ \sqrt{3}\right)\ \ \text{F(X):}\frac{2\ \sqrt{3}\ R^3\ \pi}{27} \right\} \end{array} \right\}$$

It is the second solution that we need, so enter 2 and press GET. Let's calculate the radius that the funnel will have. Press ENTER to make a copy of the solution, and then HEAD to extract:

$$\text{MIN:}\left(h = \frac{R}{3}\ \sqrt{3}\right)$$

from the list. Enter $r = \sqrt{R^2 - h^2}$ again, press ▶, then SUBST and then expand. You get:

$$r = \frac{\sqrt{6}\ R}{3}$$

The circumference of the funnel, $2\ \pi r$ is connected with the angle φ be the relation:

$$2\ \pi r = R\ ($$

Enter:

$$2\ \pi r = R$$

Now enter and press SOLVE to get:

$$= \frac{2\ \pi r}{R}$$

Press OVER , then SUBST and the EXPAND, to get:

$$= \frac{2\ \sqrt{6}\ \pi}{3}$$

This is the angle of what we use to make the funnel, i.e. the angle of the piece that we cut off is the rest:

$$2\ \pi - \frac{2\ \sqrt{6}\ \pi}{3}$$

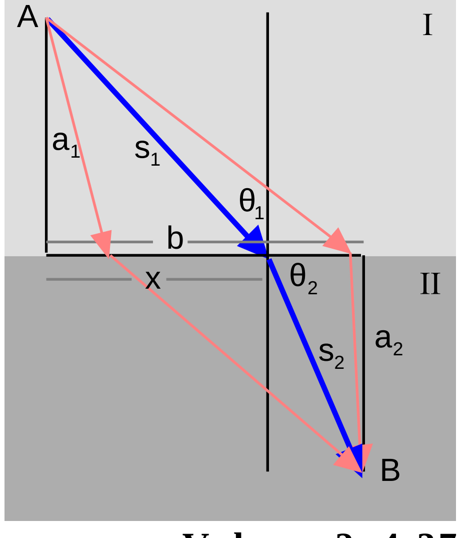
Press ENTER to make a copy of the expression, then EQ→ to separate the left from the right hand side of the equation, $\frac{2\ \sqrt{6}\ \pi}{3}$.

Press NIP to get rid of the on stack level 2. Enter $2\ \pi$, press ▶ and then -. Expand the expression to get:

$$\frac{\left(6 - 2\ \sqrt{6}\right)\ \pi}{3}$$

This is the angle of the piece to be cut out. If you want to convert this to degrees, then press R→D to get: 66.0612308665. This result is meant as decimal degrees. If you want to convert it to degrees, minutes and seconds, press now →HMS. The result, 66.0340431119 is meant as: 66°03'40''431119 . Enter now R and press UNASSUME to remove all assumptions about R , and press ◀ to drop R from the stack.

We continue with an example from physics. Some phenomenon, be it radiation, sound, or whatever, propagates itself from medium I to medium II. The propagation velocity in medium I is v_1 and in medium II is v_2. We want a relation between the angles θ_1 and θ_2. That means that we ask: Assume that the phenomenon reaches at the separation surface between the two media at an angle θ_1. What will be the angle θ_2 when the phenomenon leaves the separation surface and continues its journey in medium II? Without further assumptions it is not possible to say anything more about the angles θ_1 and θ_2. The phenomenon can choose any possible way (red feathered lines). We have to assume something, then follow the



mathematical consequences of the assumptions, make predictions, and prove them experimentally. We make the assumption that the phenomenon will choose the way that minimises the time to go from A to B. This enables us to find which of the all possible ways the phenomenon will choose (blue bold lines). According to Pythagoras we have:

$s_1 = \sqrt{a_1^2+x^2}$ and $s_2 = \sqrt{a_2^2+(b-x)^2}$

The time t_1 needed by the phenomenon to cover the distance s_1 in medium I is:

$$t_1 = \frac{\sqrt{a_1^2+x^2}}{v_1}$$

The time t_2 needed by the phenomenon to cover the distance s_2 in medium II is:

$$t_2 = \frac{\sqrt{a_2^2+(b-x)^2}}{v_2}$$

The time t for the sum of the two distances is:

$$t = \frac{\sqrt{a_1^2+x^2}}{v_1} + \frac{\sqrt{a_2^2+(b-x)^2}}{v_2}$$

The above expression is a function of x. Varying x we can cover all possible ways that the phenomenon can take. The quantities a_1 and a_2 remain (can be hold) constant when we vary x. We can find a minimum for t, if we consider it as a function of x. We have to find the roots of the first derivative of this function. Enter:

$$\frac{\sqrt{a1^2+x^2}}{v1} + \frac{\sqrt{a2^2+(b-x)^2}}{v2}$$

then enter x, and then press $\boxed{\partial}$. The HP49G returns:

$$\frac{v1\,\dfrac{2\,x}{2\,\sqrt{a1^2+x^2}}}{SQ(v1)} + \frac{v2\,\dfrac{2\,(b-x)\,-1}{2\,\sqrt{a2^2+(b-x)^2}}}{SQ(v2)}$$

Instead of solving for x we consider the following: The quantity $\sqrt{a_1^2+x^2}$ is the distance s_1. Enter the list $\left\{\sqrt{a1^2+x^2}\quad s1\right\}$ and press $\boxed{\uparrow MATCH}$ to get:

$$\frac{v1\,\dfrac{2\,x}{2\,s1}}{SQ(v1)} + \frac{v2\,\dfrac{2\,(b-x)\,-1}{2\,\sqrt{a2^2+(b-x)^2}}}{SQ(v2)}$$

on stack level 2 and a 1. on stack level 1. Drop the 1. form the stack. Similarly the quantity $\sqrt{a_2^2+(b-x)^2}$ is the distance s_2. Enter the list $\left\{\sqrt{a2^2+(b-x)^2}\quad s2\right\}$ and press $\boxed{\uparrow MATCH}$ to get:

$$\frac{v1\,\dfrac{2\,x}{2\,s1}}{SQ(v1)} + \frac{v2\,\dfrac{2\,(b-x)\,-1}{2\,s2}}{SQ(v2)}$$

on stack level 2 and a 1. on stack level 1. Drop the 1. form the stack. Press $\boxed{\blacktriangledown}$, select the sub expression:

$$\frac{v1\ \frac{2\ x}{2\ s1}}{SQ(v1)}$$

and press EXPAND to convert it to:

$$\frac{x}{v1\ s1}$$

Now select the sub expression

$$\frac{v2\ \frac{2\ (b-x)\ -1}{2\ s2}}{SQ(v2)}$$

and press EXPAND to convert it to:

$$\frac{x-b}{v2\ s2}$$

Press enter to put the edited expression to the stack, which now is:

$$\frac{x}{v1\ s1}+\frac{x-b}{v2\ s2}$$

From the picture on page 1-37 we see that $x = s_1\ \sin(\theta_1)$ and that $b = x + s_2\ \sin(\theta_2)$. Enter $b = x + s2\ \mathrm{SIN}(\theta 2)$ and press SUBST to get:

$$\frac{x}{v1\ s1}+\frac{x-(x+s2\ \mathrm{SIN}(\theta 2))}{v2\ s2}$$

Press ▼, select the sub expression:

$$\frac{x-(x+s2\ \mathrm{SIN}(\theta 2))}{v2\ s2}$$

and expand it to convert it to:

$$-\frac{\mathrm{SIN}(\theta 2)}{v2}$$

Press ENTER to put the edited expression to the stack, which now is:

$$\frac{x}{v1\ s1}-\frac{\mathrm{SIN}(\theta 2)}{v2}$$

Now enter $x = s1\ \mathrm{SIN}(\theta 1)$ and press SUBST to get:

$$\frac{s1\ \mathrm{SIN}(\theta 1)}{v1\ s1}-\frac{\mathrm{SIN}(\theta 2)}{v2}$$

Press ▼, select the sub expression:

$$\frac{s1\ \mathrm{SIN}(\theta 1)}{v1\ s1}$$

and expand it to convert it to:

$$\frac{\mathrm{SIN}(\theta 1)}{v1}$$

Press ENTER to put the edited expression to the stack, which now is:

$$\frac{\mathrm{SIN}(\theta 1)}{v1}-\frac{\mathrm{SIN}(\theta 2)}{v2}$$

This is the second derivative of the propagation time t for variable x

expressed as a function of the angles θ_1 and θ_2. It must be equal to 0 for a minimum of time. Enter 0 and press $\boxed{=}$ to obtain:

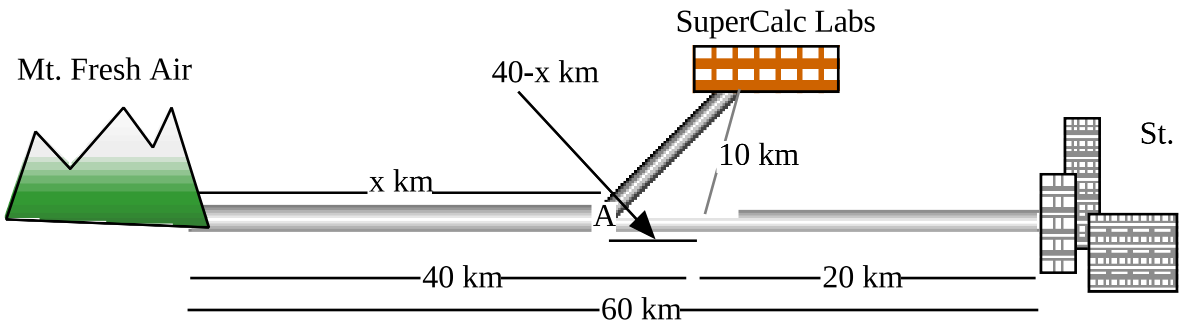
$$\frac{SIN(\theta 1)}{v1} - \frac{SIN(\theta 2)}{v2} = 0$$

which of course is equivalent to the refraction law of Snellius:

$$\frac{SIN(\theta 1)}{v1} = \frac{SIN(\theta 2)}{v2}$$

The results of experiments made for a huge number of media pairs agree very well with the above law. The assumption that we made about the shortest possible way of propagation (i.e. shortest time) seems to be very good. Indeed, it seems that this is a general basic principle of nature, to take the shortest "easiest" way. This is something that has been proved and checked and examined in hundreds and thousands of experiments, and so we simply assume a general principle to be existent. It is *not* a result of deduction, as it doesn't follow from any other assumptions (axioms) that are more simple. The only argument for accepting this principle is a huge (but still finite) number of experiments. We simply assume here that what happened many many times (in experiments) will also happen always. We draw a general conclusion out of many particular experiments, i.e., we conclude for an infinite number of experiments out of a finite number of experiments by induction. This induction is not the same as perfect induction in mathematical proof. It is imperfect induction. Nevertheless, though the word "imperfect" may have a curious taste sometimes (what does "imperfect" do in science?), the method that we followed in this example, called *inductivism*, has lead to some of the greatest discoveries. The whole building of thermodynamics, and also the foundation of the thoughts of Einstein when he started with relativity, are based on pure imperfect induction. In thermodynamics many many many experiments that failed to produce energy out of nothing, led the scientists to the assumption that this might be a general principle and that no experiment whatsoever will manage to produce energy out of nothing. Note that nobody ever can accomplish to do all possible experiments to prove this, simply because there is an infinite number of experiments that can be made. After the 10000th experiment, we simply thought: "Let's abandon experiments to produce energy out of nothing and assume that this is impossible. Let's see what follows out of this assumption". And what followed was an enormous theory that covers a wide field of phenomena. The discoveries that followed were always in agreement with the principle: "You can't produce energy out of nothing". So the theory - thermodynamics - is one of the most stable buildings of human thought, though it is not strictly proven by deduction - i.e. nobody can exclude the possibility that Rcobo will make some machine for producing energy out of nothing (and be declared to public enemy number one by the CEOs of oil producing/selling companies ;-)) Same with relativity. After so many experiments that failed to prove the existence of the ether (the assumed medium that carries light waves), everybody kept on changing the assumed properties of the assumed medium, so that the experiment that just failed, shouldn't be able to prove the existence of ether at all. (Fail first, then explain the failure… and try again !!!) Einstein simply thought: "Let's assume, the darn thing doesn't exist. What follows out of this assumption?". And what followed was one of the most impressive (and sad) chapters of science. Unbelievable predictions were experimentally proved and found to be correct. Our whole picture of the world changed dramatically. You see how it goes in this philosophical direction. Many experiments with the same result lead to the assumption that all (!) possible experiments will also have the same result. The scientific world "smells" the presence of a general principle. The assumed principle is then taken as an axiom, though it might be much more complex than mathematics axioms, and the mathematical consequences of this axiom are followed. Working on this we discover "laws" which are provable and can be *falsified* experimentally. We make the experiments and compare the results to the predictions of the "laws". If there is no agreement, we must accept that the assumed principle was either wrong or at least not perfectly conceived. If there is agreement, well… then we can use the theory and the principles as long as Rcobo doesn't sell his energy producing machine. But still, nobody can assure us that the "laws" are really that valid. There is always the possibility of another set of assumptions which leads to the same predicted results as our "laws" do, but otherwise is completely different from what we assume to be a general principle. It is only the immensely big number of successful experiments that make us assume that the principle is always valid.

We continue with another example, which is somehow more "everyday life". The inhabitants of St. Concrete City have wasted all the water that was available around the city washing their cars, going under the shower at least ten times every single day, and using it also for otherwise "wise chosen" purposes. When they realised that no water was available in the neighbourhood any more, they started complaining, what government was this that left people without water. (Note: As always, it is not "we" that did the mistake, oh no, it was the government - stupid humans - the same creatures that demanded protection of environment were wasting water until no drop was available to drink.) The atmosphere in the city was very explosive. The smallest spark would suffice to cause a detonation. Then somebody discovered that about 60 kilometres to the west there was a big amount of good water hidden in a cavity under Mt. Fresh Air. Since humans tend to believe that everything out there is under their possession and that they have the God given right to demand everything (though God might not even exist) they automatically considered the water under the mountain as yet another resource to waste. Some environmentalists talked about possible plans to reduce water consumption so that nature would have the time to refill that cavity, but who cares about what comes after us, if we can live in a barrel of waste until we leave this world? He, he, and so the administration started planning how to transport the water to the city. The one and only factor that they considered was: It has to be cheap! Oh yes, for them the world is a set of things (that of course belong to us automatically) and each and every thing has an adhesive label with its price. The distance from St. Concrete City to the waters of Mt. Fresh Air was 60 kilometres, as said above. Between the city and the water there was a laboratory of Super Calc Corp., where water had to be supplied too. It was in a distance of 20 kilometres to the west and 10 kilometres to the north of the city. The water should be transferred through pipelines to the city and to the laboratory. Since the amount of water that would flow through the tubes was different at different segments of the pipelines, the administration decided that the tubes of the whole system would also be different from each other. The tubes for the segment from the water to point A were the most expensive at 30 Solars per meter. (They named their money "Solar" to induce a connection to the sun - nature -, so that the thirsty non-thinking population had a peaceful sleep, because everybody assumed "we do protect our environment".) The tubes from point A to the laboratory had a cost of 12 Solars per meter. And the tubes from point A to the city were at 22 Solars per meter. Note again that no other specification was given in construction plans of the economists. The whole world is for them a question of costs. What a beautiful model of the world, a single variable is enough to describe everything! Poor mathematicians and physicists that still search for the truth, when the truth is known to be God and the administrative model of the universe ;-) Having all these data, the administration started trying to solve a really difficult problem. How big has the distance from Mt. Fresh Air to point A to be (call it X), in order to achieve cost minimisation? Tremendous scientists as they are, they started "putting the numbers" and calculating the costs for each and every possible length x. Of course the number of possible cases is a bit too much to be calculated in a reasonable amount of time, and because even time is money (a great economy axiom), the wise administration people put themselves in a trap. On the one hand their mathematics skills



are, well, at about the same level as that of the cows of Trabakoulas[1], but on the other hand they insist wanting to solve the problems of the world (which, by the way, they create themselves.) After some months of planning (i.e. "putting the numbers") they suspected that there might be another way to solve the problem, so that they can go playing golf with their friends, the lawyers. They had heard of some strange people out there, who were able to solve problems by writing strange letters and magic formulae on paper. (And by thinking, but this concept was not known by the administration.) So, they opened a telephone book and searched under "Mathematicians, physicists, and other strange people". They found Prof. Matt o'Mathew, an unemployed mathematician (he didn't have expensive suits), called him and arranged a meeting for describing the "very hard" problem that they had to solve. Mr. o'Mathew came to the meeting (some minutes too late - you see he was not a "serious scientist"), and the economists looked at him from his head to his toes, with a very examining expression in their faces (he had blue jeans and a T-shirt on). They described their problem, Mr o'Mathew looked at them with disbelief, and said:

- Let me see if I understood that right. You are not able to solve a the problem that my parrot is able to solve? And you want me to help you?

- Errh, yes, well… we just wanted you to prove if we… put the right numbers. Of course your efforts will be honoured adequately, with 1000000 Solars… is that enough?

- Mwahahahah - what a joke! For such a kindergarten problem, I don't take money, mwahahaha, a real puzzle you said… mwahahaha help me my stomach is aching.

When Mr o'Mathew could breath again, he did what we are going to do. He used plain calculus. And had the grace to (try to) explain what he did to the administration. (And also to the lawyers, who were present and ready to accuse Mr. o'Mathew later because of possible "wrong" solution.)

- Let the unknown distance be called X. Then…

- Wait! What is X? Is that where… errhhh, we put the numbers?

- This, dear unalphabetised human, is a label. A representative of all possible distances.

- Oh no, we don't want all possib…

- All possible distances, out of which we are going to pick that one, which minimises the costs.

- Oh no, we want…

- Shut up!

- OK.

- The costs for the segment from Mt. Fresh Air to point A are then given by 30 X

- Huh?

- Thirty dollars per kilometre times X kilometres, this will be the cost. Got that?

- Mhhh… but X is not a number and so…

- And so nothing is wrong. The costs of the pipeline for the segment from point A to the city consist of two parts. One of these parts is the distance of 20 kilometres pipeline with a price of 22 Solars per kilometre. This is 20 22.

- Ah, I understood that!

[1] Where the cows at least can solve the equation "Eat as long as you are hungry but don't eat up the universe". And, oh yes, there *were* economists around, who cared more for mathematics than for expensive suits, but they were ignored, since they were not "serious scientists" in their blue jeans and T-shirts.

- Congratulations for your mental powers! The other part is $(40-x)\ 22$, as we can see on the picture.
- Errh! What is…
- Sigh! Don't bother understanding it. Last thing we have is the costs for the tubes for the laboratory. The distance from point A to the laboratory in kilometres is given by:

$$\sqrt{(40-x)^2+10^2}$$

- Gasp!
- So that the sum of costs is given by:

$$30\ x+20\ 22+(40-x)\ 22+12\ \sqrt{(40-x)^2+10^2}$$

- …
- This is a function of a single variable, x. We can find its minimum by finding the roots of:

$$\frac{\partial}{\partial x}\left(30\ x+20\ 22+(40-x)\ 22+12\ \sqrt{(40-x)^2+10^2}\right)=0$$

Some administrators lost their consciousness at this point, Mr. o'Mathew smiled with an expression of satisfaction and the lawyers prepared the trial against the mathematician because of… planned psychological pressure and injury.

Mr. o'Mathew entered:

$$30\ x+20\ 22+(40-x)\ 22+12\ \sqrt{(40-x)^2+10^2}$$

and then x in his HP49G. He pressed ENTER to make a copy of the expression for later, then he pressed ∂ and got:

$$30+22\ \ -1+10\ \frac{2\ (40-x)\ \ -1}{2\ \sqrt{(40-x)^2+10^2}}$$

He entered x and pressed SOLVE, and after some seconds he got the error:

Not reducible to a rational expression.

He looked at the administrators that were mentally (more or less) present, and asked them:

- You had your fingers in the development and production of this machine, didn't you?

The administrators looked each other, wondering how mathematics can help somebody to find out such top secrets. The lawyers started looking the administrators with a sinister smile. They smelled another trial against the administrators that helped manufacturing such a machine.

Mr. o'Mathew went the dangerous way. He dropped the x from the stack and pressed OBJ→. He dropped the $+$ and the 2, pressed +/- and then = to get:

$$30+22\ \ -1=-\left(10\ \frac{2\ (40-x)\ \ -1}{2\ \sqrt{(40-x)^2+10^2}}\right)$$

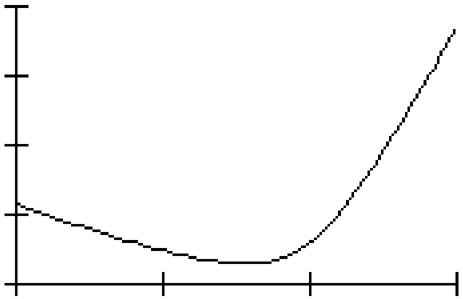
Then he pressed x^2 and EXPAND to get:

$$64=\frac{100\ x^2-8000\ x+160000}{x^2-80\ x+1700}$$

At this point he entered x again and pressed SOLVE. After some seconds the HP49G told him:

$$\left\{ x = \frac{160}{3} \quad x = \frac{80}{3} \right\}$$

Of these two solutions only one would minimise costs. So he pressed ▶ to swap stack levels 1 and 2, pressed STEQ and plotted the expression for $x = 0$ to $x = 60$ using autoscaling. The graph had a minimum at about $x = 30$. He moved the graphics cursor near $x = 30$, pressed FCN and then EXTR . After some seconds the HP49G displayed a minimum at $x = 26.6666666667$. The costs there were 1700 Solars. It is the second solution, $X = \frac{80}{3}$, which minimises costs. He pressed CANCEL to return to the stack, looked at the well dressed economists and lawyers and said.



- You will have the minimal costs if you fork the pipeline at a distance of about 26.6666666667 kilometres from Mt. Fresh Air.

The lawyers looked at him with anger and said:

- We don't pay you for results that are not exact. We want exact results. You didn't…

- For your information you don't pay me at all. And if you want exact results, here you are: $\frac{80}{3}$

- This is not a numb…

- Are you sure about it? Never heard of rationals?

- Errhmm…

- Now, decide what you want from me. A number that you can comprehend using the brown cell that you use as brain or the exact result.

- Ahem… How did you find this result? How can we be sure? You have to prove that this is really the solution.

- Well, I just did. The fact that you didn't understand anything shows me that nobody paid attention to what you did when you went to school. I suggest you to prove that there is a solution that makes costs even smaller.

The joined forces of lawyers, administrators and other unalphabetised personnel are still trying to find a better solution. They are still "putting the numbers". They still hope to find a better solution, accuse Mr. o'Mathew of betraying them, influence the jury (another set of independent ignorants) using such "proven facts" like the non-serious outfit of Mr. o'Mathew, and charge him with some millions of Solars for telling the truth.

And we, the few that are interested for the truth, can only sit and watch that on tv. Or send the jury, the lawyers and the administrators back to school. With Hulk as teacher of course.

Let's move on to the next example, which takes us out there somewhere in the universe. You remember of course the example of satellites with almost square orbits that we had in volume 1 of the Basic Calculus Marathon. We continue this investigation here, with the question: Under which conditions will the satellite have an orbit that is as similar as possible to some regular polygon? Let's first recall what we have found in the first volume of the Basic Calculus Marathon, when we examined the orbit of the satellite of a planet around the star. The coordinates x

and y of the satellite are given by:

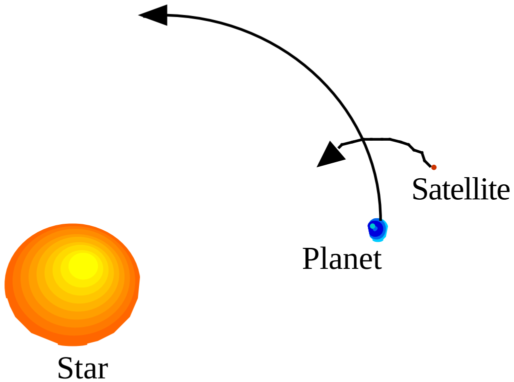
$$x = R\ \cos(\Omega\ t) + d\ \cos(\omega\ t)$$
$$y = R\ \sin(\Omega\ t) + d\ \sin(\omega\ t)$$

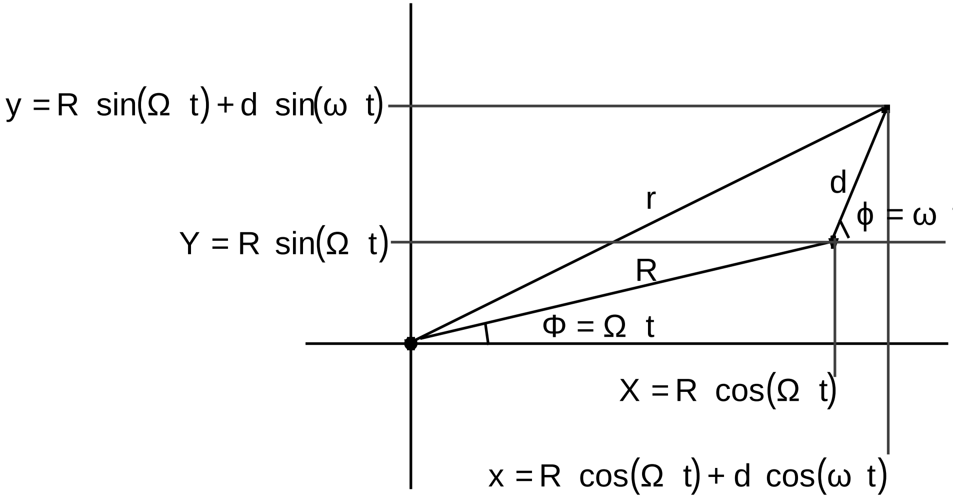
where R is the distance from the planet to the star, d is the distance from the planet to the satellite, Ω is the circular velocity of the motion of the planet around the star ω is the circular velocity of the motion of the satellite around the planet, and t is time. Enter the expression:

$R\ COS(\Omega\ t) + d\ COS(\omega\ t)$

and store it in x. (Small letter.) Enter the expression:

$R\ SIN(\Omega\ t) + d\ SIN(\omega\ t)$





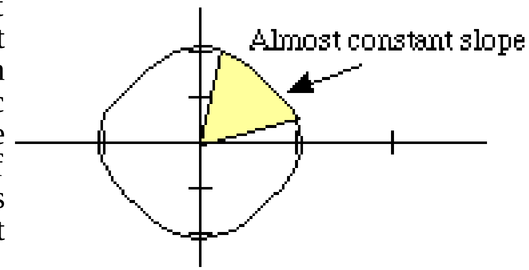
and store it in y. The distance R is much greater than d. Let's define the ratio $\alpha = \dfrac{R}{d}$, where α is real and positive, to replace R by $\alpha\ d$ in the above formulae. Enter $R = \alpha\ \ d$. Similarly, we can define $n = \dfrac{\omega}{\Omega}$, where n is positive integer. It has to be an integer, because we want to find the conditions under which the orbit of the satellite is a regular polygon. Enter $\omega = n\ \Omega$. Now, enter $\{\ \}$ (empty list) and press $\boxed{+}$ twice to get $\{R = \alpha\ \ d\ \ \ \omega = n\ \ \Omega\}$. The quantity $\Omega\ \ t$ is equal to Φ, the angle of the planet in its circular motion around the star. Enter $\Omega\ \ t =$ and press $\boxed{+}$ to get $\{R = \alpha\ \ d\ \ \ \omega = n\ \ \Omega\ \ \ \Omega\ \ t = \ \ \}$. Store this list in SUBSLST. Recall x to the stack, recall SUBSLST, enter 1 and then the program `<< SUBST >>`. Press **DOSUBS** and then **EXPAND** to get:

$d\ \ COS(\ \ \ n) + \alpha\ \ d\ \ COS(\ \ )$

This is the coordinate x of the satellite written in terms of our new variables. Store it in x1. Now, recall y to the stack, recall SUBSLST, enter 1 and then the program `<< SUBST >>`. Press **DOSUBS** and then **EXPAND** to get:

$\alpha\ \ SIN(\ \ \ n) + \alpha\ \ d\ \ SIN(\ \ )$

This is the coordinate y of the satellite written in terms of the new variables. Store it in y1. Now let's take a look at the almost square orbit that we plotted in volume 1 of the Basic Calculus Marathon. We can see that the slope of the orbit remains constant or almost



constant at some intervals of Φ, while it changes rapidly in other regions. We will try to find out, for which values of α and n the slope is as constant as possible. That means, that we have to find out the first derivative $\frac{\partial y}{\partial x}$ of the parametric function:

$$x = d\ \cos(\Phi\ n) + \alpha\ d\ \cos(\Phi)$$
$$y = \alpha\ \sin(\Phi\ n) + \alpha\ d\ \sin(\Phi)$$

as a function of the parameter Φ.

Recall y1, then x1, enter and press dF1F2 to get:

$$-\frac{n\ COS(\Phi\ n) + \alpha\ COS(\Phi)}{n\ SIN(\Phi\ n) + \alpha\ SIN(\Phi)}$$

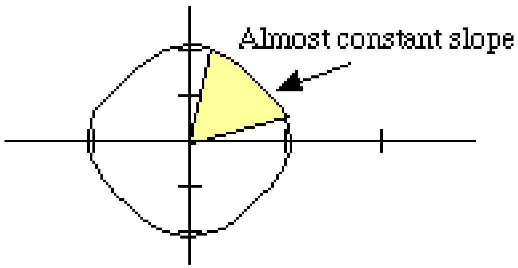
This is the first derivative $\frac{\partial y}{\partial x}$ as a function of the parameter Φ. Store it in der1. When the first derivative (slope) is as constant as possible, then the second derivative is "as zero as possible". Let's find the second derivative with respect to Φ. Recall der1, enter , press $\boxed{\partial}$, TCOLLECT and then EXPAND to get:

$$-\frac{(2\ \ \alpha n^2 + 2\ \ \alpha n)\ COS(\ \ n-\ \ ) + 2\ n^3 + 2\ \ \alpha^2}{\left(2\ \alpha\ n\ COS(\ \ n+\ \ ) - \left(2\ \alpha\ n\ COS(\ \ n-\ \ ) - \left(n^2\ COS(2\ \ \ n) + \alpha^2\ COS(2\ \ \ ) - (n^2 + \alpha^2)\right)\right)\right)}$$

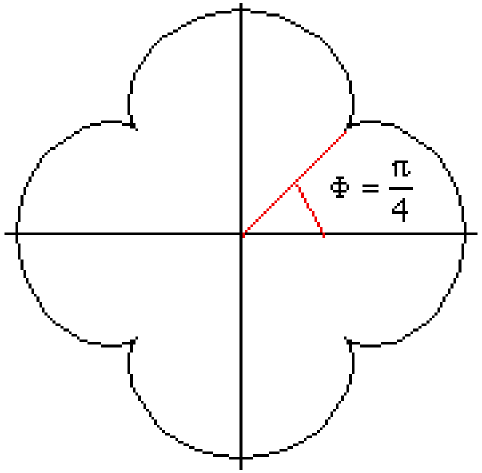
Store this expression in der2. This is the second derivative with respect to Φ. Now we want to find for which values of α and n this derivative is equal to 0 for certain intervals of Φ. Notice that the numerator of the above ration is quadratic in α. We can solve it for α and if the denominator is different from 0 for these values of α, then we have found corresponding pairs of n and α that make the orbit of the satellite look like a polygon at the specified range of angle Φ. Recall der2, enter α and press SOLVE. The HP49G returns:

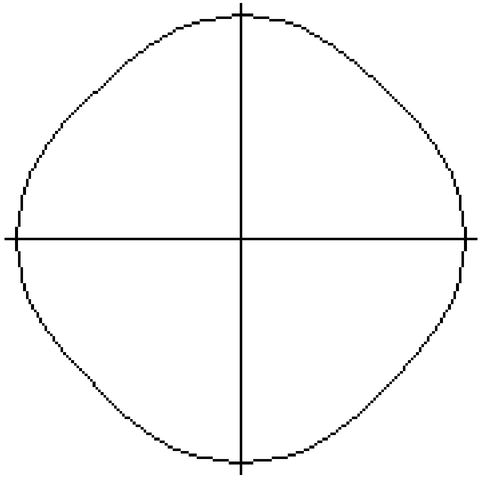
$$\left\{ \begin{array}{l} \alpha = -\dfrac{\left(COS(\ \ n-\ \ )\ n^2 + COS(\ \ n-\ \ )\ n - n\sqrt{COS(\ \ n-\ \ )^2\ n^2 + (2\ COS(\ \ n-\ \ )^2 - 4)\ n + COS(\ \ n-\ \ )^2}\right)}{2} \\ \alpha = -\dfrac{\left(COS(\ \ n-\ \ )\ n^2 + COS(\ \ n-\ \ )\ n + n\sqrt{COS(\ \ n-\ \ )^2\ n^2 + (2\ COS(\ \ n-\ \ )^2 - 4)\ n + COS(\ \ n-\ \ )^2}\right)}{2} \end{array} \right\}$$

We see that there are 2 solutions. Each of them corresponds to a physical situation that we are going to examine now. Store the solutions list in SOLα. Lets start with the case of an almost square orbit. In this case we have $n = 5$, i.e. $\omega = 5$. The orbit of the satellite is almost "a line" at $\Phi = \frac{\pi}{4}$. Store 5 in n and $\frac{\pi}{4}$ in . Recall SOLα and expand. The result is $\{\alpha = 5\ \ \ \alpha = 25\}$. That means that for $n = 5$ and $\alpha = 5$ or $\alpha = 25$, the orbit gets similar to a square. It is "mostly similar" to a square at $\Phi = \frac{\pi}{4}$ (and also at $\Phi = \frac{\pi}{4} + \frac{\pi}{2}$, $\Phi = \frac{\pi}{4} + \frac{2\ \ }{2}$, $\Phi = \frac{\pi}{4} + \frac{3\ \ }{2}$). Let's try some orbit plots using these values. Recall x1 and y1. Multiply y1 by i and add the result to x1. Store the result in ORBIT. The expression stored in orbit contains also the quantity d, which is the distance from the planet to the satellite. No matter how big this distance is, the shape of the orbit will remain the same. It will only



be scaled according to d. Since we are interested for the shape but not the size of the orbit, store 1 in d. Let's try a plot with the first solution for α. Store 5 in α. Now set approximate mode, go to the PLOT SETUP screen, choose plot type PARAMETRIC, enter 'ORBIT' in the input field EQ:, and ' ' in the input field Indep:. (Both with quotes.) Go to the PLOT WINDOW – PARAMETRIC screen, set horizontal view from −13 to 13, vertical view from −6.5 to 6.5, Indep Low: to 0., High to 6.28319, and Step: to .05. Press ERASE and then DRAW. The orbit doesn't look like a square at all. In fact something rather interesting happens at $\Phi = \frac{\pi}{4}$ (and also at $\Phi = \frac{\pi}{4}+\frac{\pi}{2}$, $\Phi = \frac{\pi}{4}+\frac{2\,\imath}{2}$, $\Phi = \frac{\pi}{4}+\frac{3\,\imath}{2}$) but we will examine this phenomenon later on. Exit the plotting environment, store 25 in α and redraw. Now you get the almost square orbit. If you play with different values for α you will see that the best possible approximation to a square orbit is for α = 25. All other values makes the orbit less square like. We found that for:

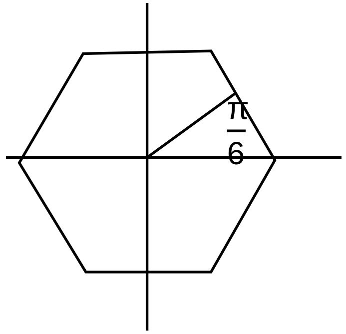




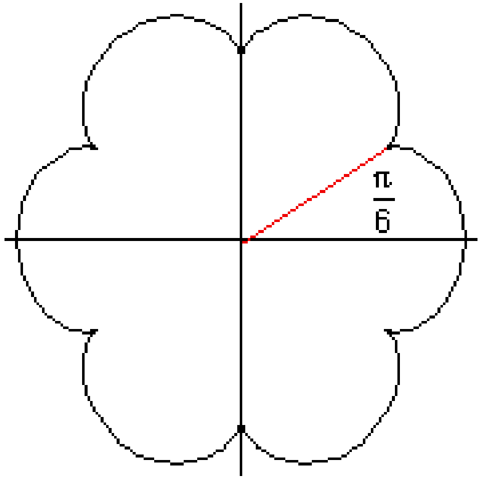
R = 25 d and ω = 5 ς

the orbit gets "as quadratic as possible".

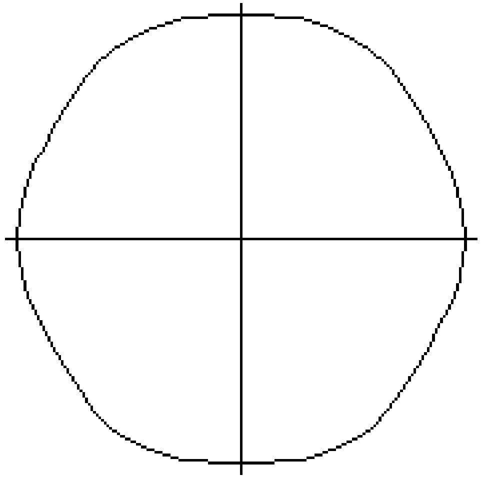
Let's try to produce an almost regular hexagon. In this case the orbit will be almost "a line" at $\Phi = \frac{\pi}{6}$. Switch to exact mode, store $\frac{\pi}{6}$ in and then store 7 in n. Purge α. Recall SOLα and expand to get:



{α = 7 α = 49}



Store 7 in α and draw again to get the "flower" orbit with the peculiar behaviour at $\Phi = \frac{\pi}{6}$ (and $\Phi = \frac{\pi}{6} + m\frac{\pi}{3}$). You must zoom out to see the whole orbit. We will discuss these peculiarities later on. Store 49 in α, set horizontal view from −128. to 128., vertical view from −64. to 64. and redraw. Now you get an almost regular hexagon. We found that for:



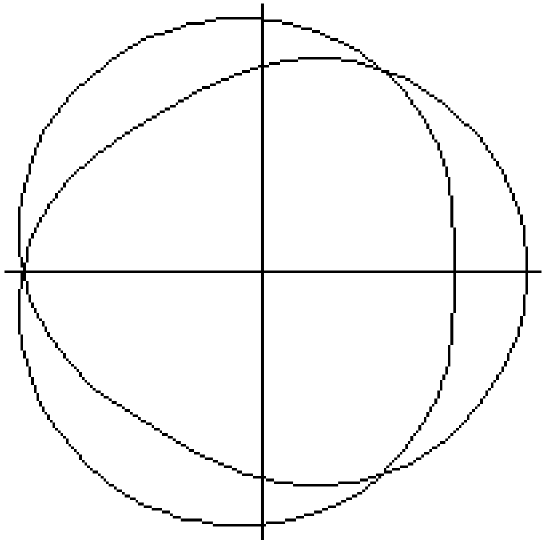
R = 49 d and ω = 7 ς

the orbit gets "as quadratic as possible".

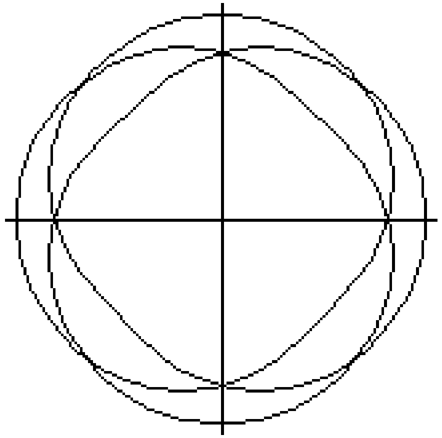
The orbit seems to get as similar as possible to a regular m-gon when:

$n = m+1$ and $\alpha = (m+1)^2$

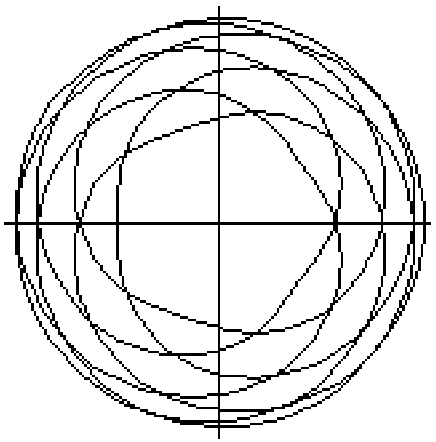
Is this a general behaviour? Try the case of a regular decagon, i.e. $n = 11$, and $\alpha = 121$. It seems that we really have a general behaviour pattern. We used both empirical and analytic thoughts to find this result, so we can't call our method pure deduction, but nonetheless it seems to be OK. Now, what if we don't limit our thoughts to regular polygons? What if we would allow broken values for n? If we allow n to be for example $\frac{5}{2}$, then the orbit will "close" after the satellite has covered an angle of $2\cdot 2\pi$ and not after 2π. The corresponding value of α would be $\left(\frac{5}{2}\right)^2 = \frac{25}{4}$ in this case. Store $\frac{5}{2}$ in n, $\frac{25}{4}$ in α, set horizontal view from -16 to 16 and vertical view from -8 to 8. Set also Indep θ Low: to 0, High to 12.6 (which is approximately 4π) and Step: to $.125$.. Now redraw the plot. The orbit looks rather different now. From the relation $n = m+1$ we get $m = n-1$. Since $n = \frac{5}{2}$ in this case, we obtain: $m = \frac{3}{2}$. So the orbit is almost a regular… 1.5-gon. Or better, a triangle whose 3 angles are on the circumference of a circle

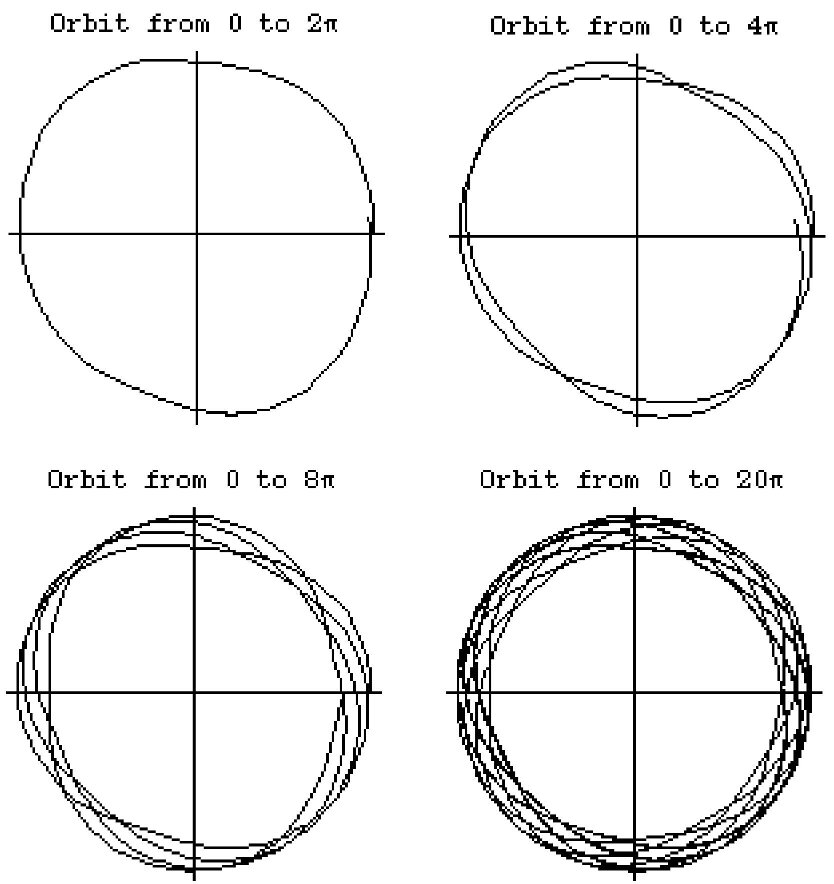


and are equidistantly "distributed" over 4π instead of 2π. Let's have another example for such broken values of α and n. Plot for $\alpha = \frac{49}{9}$ and $n = \frac{7}{3}$. The independent variable θ goes from 0 to $3\cdot 2\pi = 6\pi \approx 18.9$ in steps of $.189$. The plot looks like the picture to the right. In this case we have $m = n-1 = \frac{7}{3} - 1 = \frac{4}{3}$, i.e. almost a regular $\frac{4}{3}$-gon, that is a square whose four angles are "distributed" over $3\cdot 2\pi = 6\pi$.



Imagine the surprise of extraterrestrials if the humans put a satellite in such an orbit around the earth! If those extraterrestrials have pattern recognition units in their brains that are similar to ours, they will ask themselves what the heck is going on, since such orbits are quite unusual in nature. Producing such orbits is like a huge light advertisement saying that intelligent (or almost intelligent) life forms are on this planet. (Or that God is playing spirograph ;-)) Take a look at the orbit with $n = \frac{12}{7}$ and $\alpha = \frac{144}{49}$. The independent variable θ goes from 0 to $7\cdot 2\pi = 14\pi \approx 44$ in steps of $.22$. Such orbits are quite difficult (if not impossible) to produce. Note that the satellite comes so close to the star, that it will eventually change orbit and become itself a planet. Nonetheless the (almost impossible) orbit has a nice shape to look.





The next step is to allow any real value for n. This will produce non closed orbits, that means that the orbit will not be strictly periodic. It will be a curve that never repeats itself. For example, for $n = \pi$ and $\alpha = \pi^2$ we get satellite orbits which are irregular curves, which are nonetheless restricted inside a ring around the star. The above plots demonstrate this.

Now that we found out empirically, that for a satellite orbit that resembles a regular m-gon (as far as possible) the relation $\alpha = (m+1)^2 = n^2$ holds, we could try to substitute α in der2 (second derivative of orbit), and solve for n. Purge the variables α, n, and . Recall der2 on the stack. The expression is a fraction. We will find the roots of the numerator and silently assume that the denominator is not equal to 0 when n is equal to a root of the numerator. Press FXND to convert the fraction to its numerator (stack level 1) and denominator (stack level 2). Press ◀ to drop the denominator. Enter $\alpha = n^2$ and press SUBST. Let's suppose that we want to find such values for n, that make the orbit similar to a square. This means that the second derivative will be 0 around $\Phi = \frac{\pi}{4}$. (Of course, if we want other polygons, we have to use other angles.) Enter $= \frac{\pi}{4}$ and press SUBST. The result of the substitutions is:

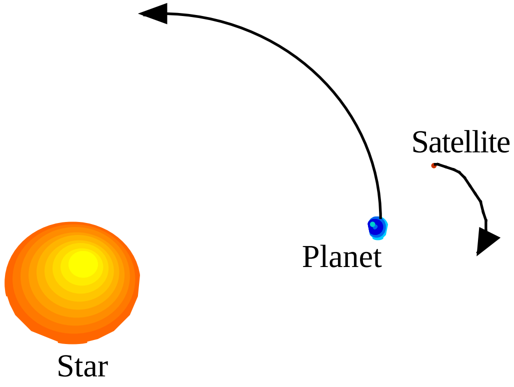
$$-\left(\left(2\ n^2\ n^2+2\ n^2\ n\right)\ \mathrm{COS}\left(\frac{\pi}{4}\ n-\frac{\pi}{4}\right)+2\ n^3+2\ \left(n^2\right)^2\right)$$

Store this in NUMERATOR. Let's try to solve this for n. Recall NUMERATOR, enter n and press SOLVE. After some seconds in agony the HP49G returns:

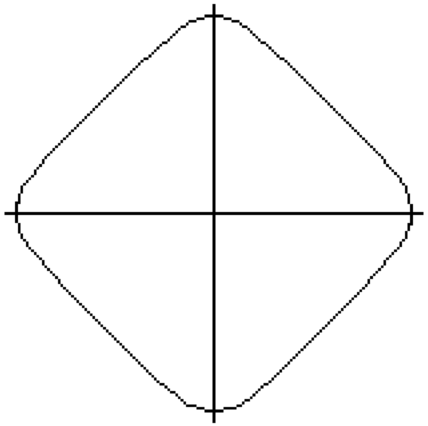
$$\left\{ n=\frac{8\ n1\ \pi+8\ \mathrm{ATAN}\left(\frac{-1}{\sqrt{2}-1}\right)}{\pi}\quad n=0\quad n=-1 \right\}$$

Store this in SOLn. Let's examine the solutions. Recall SOLn and press HEAD to extract the first solution out of the list. Take it to the EQW. The sub expression $\mathrm{ATAN}\left(\frac{-1}{\sqrt{2}-1}\right)$ can be converted to a quotient but not by expanding. Select this sub expression and press

→NUM. Then press XQ. The result is $\frac{-3}{8}$ 1. Press ENTER and then EXPAND. Now you have $n = 8\ n1 - 3$ on stack level 1. We have found that for an almost square orbit - or more precisely, for an orbit whose second derivative is 0 at $\Phi = \frac{\pi}{4}$, the possible values for n are of the form $n = 8\ n1 - 3$, where $n1$ is an arbitrary integer. Let's find some values for n. Enter $n1$ -2, 2, 1, and press SEQ. The result is the list



$\{n = -19 \quad n = -11 \quad n = -3 \quad n = 5 \quad n = 13\}$. We already know that $n = 5$ corresponds to an almost square orbit, and so we see that $n = 12$ corresponds to a dodecagon (12-gon). But what are the negative solutions? We already know that $\omega = n\ \Omega$. from this relation we see that for negative values of n the satellite will have negative angular velocity if the planet's angular velocity Ω is positive, which means that the satellite runs retrograde. Though such orbits are known to be often



unstable, let's plot such an orbit for $n = -3$. Store -3 in n, 9 in α, set horizontal view from -20 to 20, vertical view from -10 to 10, Indep Low: to 0, High to 6.29, and Step: to $.0629$. Now, draw the plot. Aha! An even better cosmic square is achieved for $n = -3$. Such negative values of n correspond to curves that resemble as far as possible regular $(1 - n)$-gons. The group of curves that the solutions of the form $n = 8\ n1 - 3$ describe have all one property in common: Their second derivative is (almost) equal to 0 around $\Phi = \frac{\pi}{4}$.

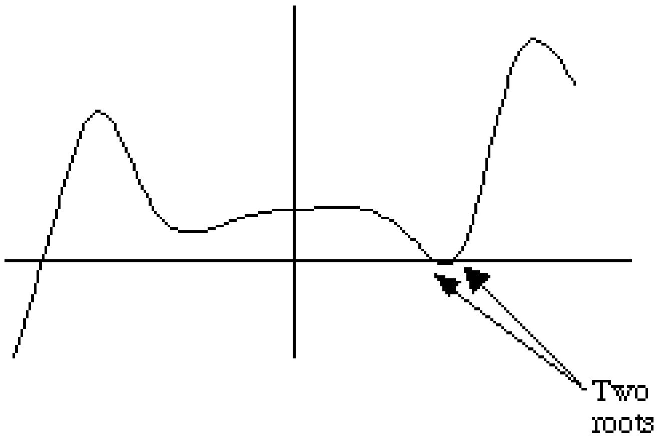
The solution $n = 0$ is a trivial solution which doesn't produce any orbit at all, i.e. the satellite "sits" at $x = 1$, $y = 0$, and the planet sits at $x = 0$, $y = 0$, which means that the planet is in the star. (Poor inhabitants - let's exclude this solution for humanity reasons ;-))

The solution $n = -1$ is also an "inhuman" one, but nonetheless interesting. It describes a planet that moves on a circular orbit around its star (planet inhabitants saved), with a satellite that oscillates with $x = 2\ \cos(\Phi)$ at $y = 0$ though the star! (Satellite and its inhabitants evaporates.)

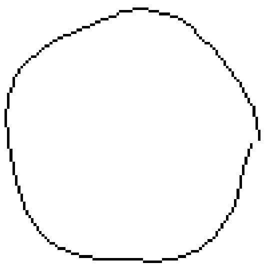
From the above we clearly see, that the mathematical solutions of a problem don't have always to describe some "real existing" system - except of course for the case of planets and satellites that withstand the conditions inside a star ;-) Mathematics seem to be "more free" than physics, it doesn't have to represent any "real world" system at all. Complete freedom of thoughts with no restrictions whatsoever. Physics is free enough for stating that "heavens is the limit". Mathematics are free enough to even wipe out heavens, and thus remove any limit. It is so free that it can prove (strictly - no "feeling based assumptions") its own imperfectness or incompleteness, its own contradictions - but that's stuff for another marathon.

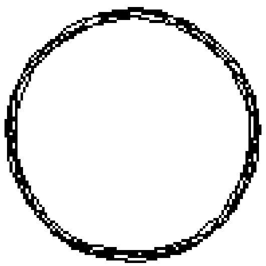
Let's make another (the last) examination of our orbits. Are still other solutions possible? And what do they represent? In the previous pages

we solved
$-((2\quad \alpha n^2 + 2\quad \alpha n)\ \cos(\quad n - \quad) + 2\ n^3 + 2\quad \alpha^2) = 0$ for n when $\alpha = n^2$ and $= \frac{\pi}{4}$. Let's plot der2 as a function of n when $= \frac{\pi}{4}$, and $\alpha = 25 = 5^2$, that is for an assumed square orbit. Store 25. in α. Enter $\frac{\pi}{4}$, press →NUM and store the result in . Now, go to the PLOT SETUP screen, select Function plot type, and enter der2 in the input field EQ:. Enter 'n' (with quotes if some value is stored in variable n). Go to the



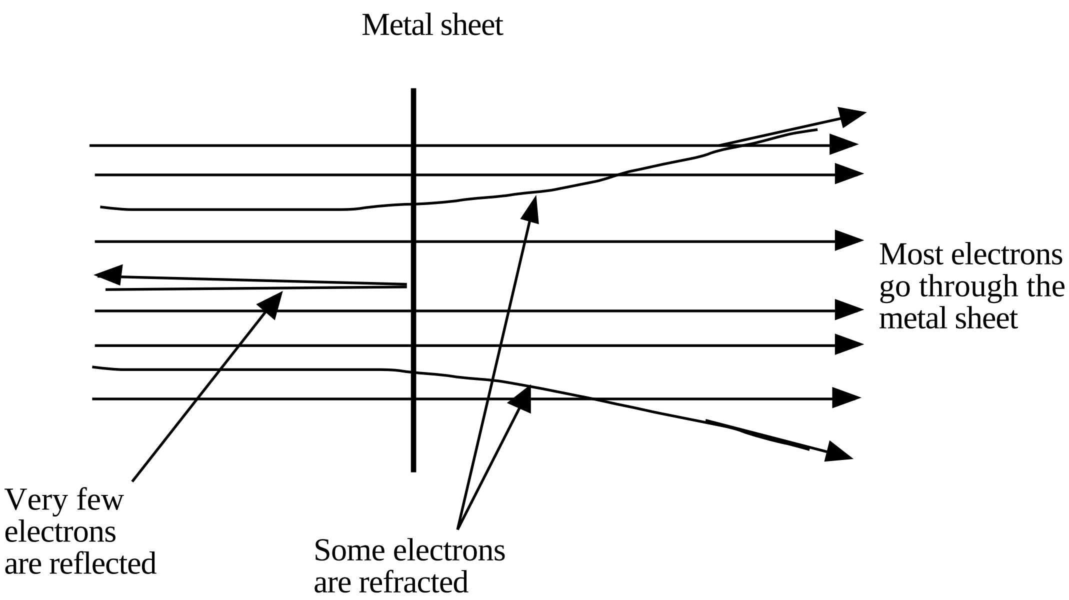
PLOT WINDOW − FUNCTION screen and enter horizontal view range from −10 to 10. use the arrow keys to select the input field Indep Low: and press RESET. You will be presented a popup menu with the options to reset all things in the screen or only the value of the current input field. Select Reset value and press ENTER.



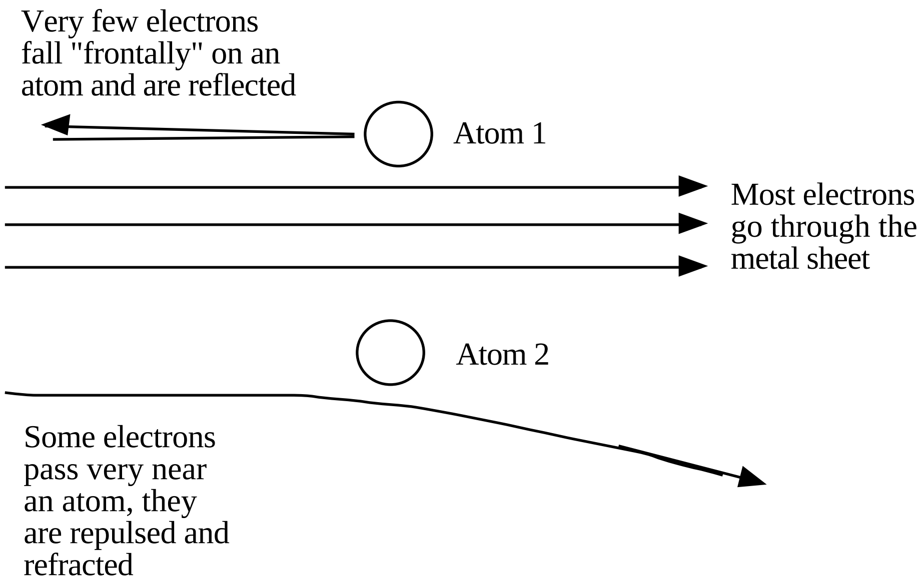


Reset also the input field Step:. Now press AUTO to autoscale the plot, ERASE and then DRAW. You can see that the function has two roots at about $n = 5$. The first of these roots is indeed $n = 5$. Move the cursor near the second root, press FCN and then ROOT. The HP49G returns the root $n = 5.7171656495$. This root describes an orbit which has its second derivative almost equal to 0 at $= \frac{\pi}{4}$ but is no regular polygon. If you make a parametric plot of ORBIT with as the independent variable from 0 to 2 you will get an open curve. If you plot with from 0 to 10 then you will get the same kind of orbit as we saw when we used $n = \pi$. These results demonstrate the fact that our condition "as similar to a regular polygon as possible" are only imperfectly described when we say that the second derivative has to be 0 at $\Phi = \frac{\pi}{4}$ or any other angle. There are also non-polygonal orbits that have this property. And some of the (real) roots that we find describe exactly these orbits. Actually we should demand that the second derivative is 0 for the range $\Phi = 0$ to $\Phi = \frac{\pi}{4}$ (in the case of a square orbit). But that would make the formulation and further work more difficult, and would presumably return no results, because this would correspond to a perfect square, which is presumably impossible. (Remember, our condition was: as similar as possible to a regular polygon, but not an exactly shaped polygon.) This also shows how assumptions and approximations are used in physics. Often, the solution of a problem in physics is very tightly related to the art of approximation by modelling and making useful assumptions. We will continue on this orbit problem at some future part of this marathon, and we will see how to we can make an even better mathematical formulation of the condition "as similar to a regular polygon as possible".

Let's turn to another orbit-like problem. We examined orbits in planetary system scale. Now we take a look at orbits in atomic scale, though we know that the word "orbit" doesn't make sense at all in these dimensions. Nonetheless, considering the used models and the history of development of quantum mechanics helps understanding many things



that have to do with our built-in pattern matching engine and the resulting naive and dangerous analogies. The first detailed picture of the atoms was constructed with the help of the experiments of Rutherford. A thin metal sheet (target) was set under bombardment by cathode rays (electrons as projectiles) and Rutherford simply looked at "what happens with the projectiles". The results clearly said that most of the projectiles simply went through the sheet! As if it wasn't there! A smaller number of projectiles changed its course and and even smaller number was reflected by the metal sheet as shown in the pictures on the next page. That was quite a surprise. It meant that what we perceive as a solid material is (at least for electrons) almost not there! How could that be? Rutherford was a physicist and not transcendental-meditative-analogy-builder. He didn't speculate, he didn't tried to guess the "laws of the universe" by postulating things that can't be proven experimentally. He simply accepted what he saw, and what he saw was definitely not what "transcendental-meditation" or religion would tell him. If most electrons simply go though, could this imply that the sheet was for its biggest part… simply full of "holes"? Was the structure of matter, not continuous? He made this assumption, but he didn't raised it to the position of a law before doing further examination of the consequences of this assumption. If matter had a "granular" structure, then the behaviour of all projectiles could be explained, no matter if they went trough or not. The "granular" structure would explain why most electrons went through. They simply didn't meet anything that could change their way or send them back. And what about the other electrons that were refracted or reflected? There had to be something that made them behave this way. Rutherford made his assumption a bit more detailed by adding that the electrons were repulsed by negative charge that had to be concentrated at certain places. The atomic units of matter had to be constructed in such a way, that negative charge (electrons) can experience a repulsive force when they pass near such a unit. There had to be negative charge "around" these units (the atoms). If so, then there had to be also positive



charge "inside" the negative charge, in order to explain why the electrons didn't get "glued" and stayed in the sheet. So the assumed atoms would be really tiny and make only a tiny part of what we perceive as solid matter, the rest of our perception being not even thin air. The atoms would have negative charge (electrons) at the outside and positive charge at the inside. Was it that way? A huge amount of experimental work has been done to prove this assumption. Many physicists of that time saw only an "artificial" assumption but not "reality" in this explanation. And it was very good to do so. Physics is not believing by simply telling. Physics is doubt, hard work, experimental proof, experimental proof of the experimental proof, and above all… curiosity that doesn't stop to ask questions when transcendental meditation declares the world to be understood completely. After all this hard work was done, the atomic assumption was accepted and the physicists asked questions about the inner structure of the atom. They found out that the atom consists of its nucleus which carries positive charge and of electrons distributed around the nucleus and carry negative charge. Experiments demonstrated that the atom can't have any possible energy but only certain values. Nothing "between" these values seemed to be allowed. This experiment also contradicts what a "meditative-transcendental-want-it-so-theory" could "deduce". One could speculate about planetary-like orbits (naive analogy) of the electrons around the nucleus, but then what about continuous energy loss because of radiation? One could also speculate about "holy places around the nucleus where the electrons just sit and do nothing, embedded in cosmic peace", but then what about the attractive forces between the negative and positive charges? Instead of doing meditation for solving physical problems, the scientists started *thinking* and having sleepless nights. Then Bohr made an assumption which seems to be like the naive analogy of planetary-like orbits, but nonetheless is way different. He postulated that of all planetary-like orbits only these were possible, in which the angular momentum of the electron is an integer multiple of $\frac{h}{2\ \pi}$, where h is Plank's CONT. That means:

$$m\ v\ r = \frac{n\ h}{2\ \pi}$$

where m is the mass of the electron, v its radial velocity around the nucleus, r the radius of its orbit around the nucleus. Why is that not a naive analogy? Well, consider our planetary system and you can immediately see that any energy is possible for a planet. There is no restriction to certain orbits that obey some rule. Of course Bohr's assumption was a bit too much, especially because he also postulated that no energy loss by radiation takes place in such "allowed" orbits. But at least it could be tested experimentally. And the experiments demonstrated that indeed his model had to do something with reality. Let's try to calculate the "radius" of the orbit of an electron around the nucleus of hydrogen, which consists of a simple proton. The energy of the electron in its assumed orbit is the sum of potential and kinetic energy. The potential energy is the energy of a positive and a negative elementary charge at a distance r from each other. For the potential energy enter:

$$-\frac{qe^2}{4\ \pi\ 0\varepsilon r}$$

where qe is the electron charge and $\varepsilon 0$ the permitivity of vacuum. For the kinetic energy enter:

$$\frac{m\ v^2}{2}$$

where m is the mass of the electron and v its radial velocity around in its assumed orbit around the nucleus. Press $\boxed{+}$ to add the potential and kinetic energy and get:

$$-\frac{qe^2}{4\ \pi\ 0\varepsilon r}+\frac{m\ v^2}{2}$$

Store a copy of this expression in H.ENERGY . Now we eliminate v

from this expression using Bohr's assumption. Enter:

$$m\ v\ r = \frac{n\ h}{2\ \pi}$$

then enter v, and press SOLVE to get:

$$v = \frac{n\ h}{2\ \pi\ r\ m}$$

Store a copy of this equation in BOHR.POSTUL. Press SUBST to substitute v in the expression for the energy and get:

$$-\frac{qe^2}{4\ \pi\ \varepsilon 0\ r} + \frac{m\left(\frac{n\ h}{2\ \pi\ r\ m}\right)^2}{2}$$

This is the total energy of the electron as a function of the radius of its assumed orbit. Since we know that phenomena in nature tend to proceed in a direction that minimises the energy of a given system, we try to find the radius r that leads to the minimum energy of the system proton-electron. That means that we want to find the roots of the first derivative of the total energy for r. Enter r and press $\boxed{\partial}$ to get:

$$-\frac{-(qe^2\ 4\ \pi\ \varepsilon 0)}{SQ(4\ \pi\ \varepsilon 0\ r)} + \frac{2\ m\ 2\ \frac{n\ h}{2\ \pi\ r\ m}\ \frac{-(n\ h\ m\ 2\ \pi)}{SQ(2\ \pi\ r\ m)}}{4}$$

This is the first derivative of the energy with respect to r. Now enter r and press SOLVE. You get:

$$r = \frac{n^2\ h^2\ \varepsilon 0}{\pi\ m\ qe^2}$$

Store a copy of this in BOHR.R. The integer n can have the values 1,2,3, . That means that the "allowed" orbits are:

$$r_1 = \frac{1\ h^2\ \varepsilon 0}{\pi\ m\ qe^2}$$

$$r_2 = \frac{4\ h^2\ \varepsilon 0}{\pi\ m\ qe^2}$$

$$r_3 = \frac{9\ h^2\ \varepsilon 0}{\pi\ m\ qe^2}$$

and so on. The first of the allowed radii is:

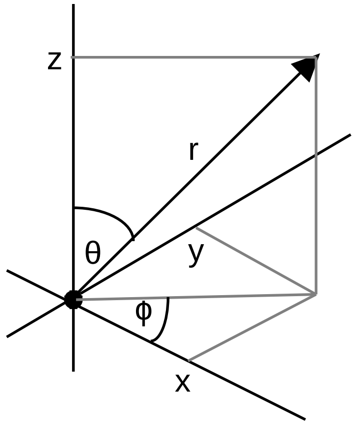
$$r_1 = \frac{1\ h^2\ \varepsilon 0}{\pi\ m\ qe^2}$$

and is called the first Bohr radius. It has the value of 52.92pm. This was verified experimentally. The results were in excellent agreement with the theory and so, at least for the hydrogen atom, the assumption of Bohr was found to be usable. Using the theoretical radius and Bohr assumption we can find the allowed energies of the electron. Recall H.ENERGY, recall BOHR.POSTUL and press SUBST. Then recall BOHR.R and press again SUBST. Expand the expression to get:

$$-\frac{m\ qe^4}{8\ n^2\ h^2\ \varepsilon 0^2}$$

These are the allowed energies of the electron of the hydrogen atom. They were confirmed by experimental work. But still, there were big problems regarding Bohr's atom model. As we know today, the reason for the problems is more or less our pattern matching recognition. In every day life we see physical objects having what we call velocity and following some particular way on their movement. So we extrapolated these observations to a world in which they might be useless, using the naive analogy: "Like a planetary system, the nucleus is like a star and the electron is like a planet". We assumed that nature would behave just as we expected, but nature doesn't have any obligation at all, to behave as we think, and above all to be understandable the way we want it to. Before we proceed to the quantum mechanical description of the hydrogen atom, a couple of (I hope destructive) words on the kings of analogy, the transcendental meditation "scientists". All transcendental meditative "theory" still bases on our perception of the world, and because our perception says that "objects have velocities", don't expect to find quantum mechanics out of singing their holy songs. Of course (ha,ha) *after* quantum mechanics was formulated and tested experimentally, those guys immediately said that it was in agreement with their transcendental meditative results without even really trying to follow the theory and its formalism by *working*. I have the feeling that whatever theory comes out and proves to be usable, they will say that they already knew it since 1483 years. Of course they say that only after other people have done the work. Thanks heavens we passed the state of the dark ages of charlatanry and nobody in the scientific world listens to such stupidities. And thanks heavens this sickness of human mind will vanish by itself.

Back to quantum mechanics. After the theory was developed, it was found out that the electron doesn't behave like an object, which is in orbit around the nucleus. The world "orbit" doesn't have a meaning in the subatomic world. At least not the meaning that we comprehend using the observations that we can make with our senses. The electron behaves more like a wave around the nucleus. (This analogy is also very dangerous but the scientists know that - we use it only for imperfect grasping of the subatomic world.) The electron is described by what we call "wave function", a function of the coordinates of the electron. It turns out that this wave function $\Psi(r,\theta,\phi)$ can be written as a product of functions of each single coordinate, i.e. it has the form $\Psi(r,\theta,\phi) = R(r)\cdot\Theta(\theta)\cdot\Phi(\phi)$. The wave function is in general a complex function. Without further interpretation we accept here that the product of the wave function with its complex conjugate is proportional to the probability to "find" the electron at r,θ,ϕ. The first wave function is fully symmetric with respect to θ,ϕ and it depends only on r, the distance from the nucleus. This wave function is:



$$\Psi(r) = \frac{1}{\sqrt{\pi}}\cdot\left(\frac{1}{a_0}\right)^{\frac{3}{2}}\cdot e^{-\frac{r}{a_0}}$$

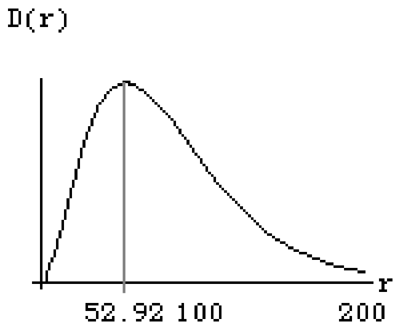
where a_0 is the first Bohr radius that we have calculated on the previous pages. The probability to observe the electron (with the above wave function) at some distance r from the nucleus is proportional to the product $\Psi(r)\cdot\Psi^*(r)$, where $\Psi^*(r)$ is the complex conjugate of $\Psi(r)$. Since this wave function is real we have $\Psi^*(r) = \Psi(r)$, and so the probability to observe the electron at some distance r from the nucleus is proportional to $\Psi(r)\cdot\Psi^*(r) = \Psi^2(r)$. The complete formula for calculating the probability to observe the electron between r_1 and r_2 is given by the integral:

$$P(r) = \int_{r=r1}^{r=r2} 4\cdot\pi\cdot r^2\cdot\Psi^2(r)\,dr$$

The part $4\cdot\pi\cdot r^2\cdot\Psi^2(r)$ is known as the radial probability distribution function $D(r)$. It is this function that gives us the probability to "see" the electron at a certain distance r from the nucleus. Let's find where the electron has its maximum probability "to be seen". Enter the radial probability distribution function of the first wave function:

$$4\cdot\pi\cdot r^2\cdot\left(\frac{1}{\sqrt{\pi}}\cdot\left(\frac{1}{a_0}\right)^{\frac{3}{2}}\cdot e^{-\frac{r}{a_0}}\right)^2$$

Now enter r and press $\boxed{\partial}$ to find the first derivative with respect to r. The roots of this derivative are the distances r from the radius, where the probability to "see" the electron takes its extremal values. Enter r, press SOLVE and wait until the HP49G returns $\{r = a_0 \quad r = 0\}$. The first solution, $r = a_0$, is the distance where we get the maximum probability. It is exactly the same like the first Bohr radius, which shows that Bohr's model was perhaps not perfect but usable! The second solution, $r = 0$, is where this probability has its minimum. This minimum probability 0, i.e. the electron will never be at $r = 0$, where the nucleus is. If you plot the radial probability distribution function:

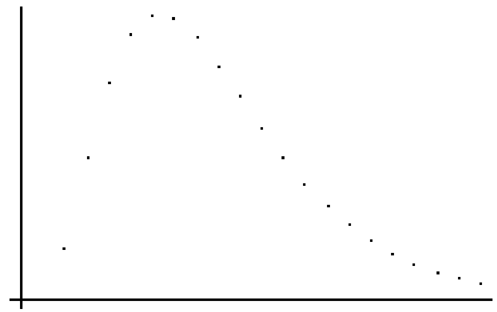


$$4 \cdot \pi r^2 \left(\frac{1}{\sqrt{\pi}} \left(\frac{1}{52.92} \right)^{\frac{3}{2}} e^{-\frac{r}{52.92}} \right)^2$$

where a_0 has been replaced by its numeric value in nanometers, then you will get the picture to the right (without the annotations).

Quantum mechanics is a fascinating chapter of physics and we are not going to examine it in much detail here. But nonetheless let's take a look at Bohr's model. Bohr didn't only assume something and raised it to a "universal law". He assumed something and followed its consequences, making predictions which were confirmed experimentally. Still, he and the whole scientific word didn't accept that "this is the absolute truth". Why? Well, first of all there can be many (in fact infinite many) assumptions and models that lead to the same results. A model is never "one and only". In case of many models that lead to the same results, further work has to be done, theoretically and experimentally, that helps us tell which one is the best. In the case of quantum mechanics, the fully developed formalism is based on assumptions (postulates) that are even harder to grasp than Bohr's assumption. We accept them because the predicted properties were confirmed in thousands of experiments, and not because we blindly believe that they are true. Furthermore quantum mechanics is a theory with which we can calculate and make predictions but the basic part of it is not "understandable" for humans. We have objects (particles) of which we think they have well defined "borders" to the "outer world", and then we find out that they are waves which end at… infinity!!! On the other hand the same particles can also behave like tiny objects with well defined dimensions, say like mini spheres. Presumably these particles-or-waves are neither particles nor waves but something else, for which we still don't have an adequate model. We can perceive physical bodies and waves with our senses and so we extrapolated these concepts (as well as possible) to a world, where they perhaps have no meaning. It could also be that these contradicting behaviours - sometimes wave, sometimes particle - have their roots in the inherently existing imperfectness/incompleteness of (almost) any formal theory. We use mathematics that include imperfectness/incompleteness, so why do we expect a perfect description of the word with no contradictions at all?

Another thing to think of: Suppose that somebody, before Bohr made his assumptions, was able to measure the radial probability distribution of the electron around the nucleus of the hydrogen atom. He or she would collect a number of $r - D(r)$ pairs. He or she could then plot these values in a scatter plot and get the picture to the right. If he or she decided to fit the data to some function of the type $C_1 \cdot r^2 \left(e^{-C_2 \cdot r}\right)^2$, where C_1 and C_2 are fitable parameters, he or she would find that with $C_1 = 2.7E31$ and $C_2 = -1.89E10$ a perfect correlation can be established.



But so what? What are $C_1 = 2.7E31$ and $C_2 = -1.89E10$? Why don't they have any other value? What does the fit function represent? Where does it come from? We see, statistics don't answer questions. *The statistical method only shows correlations but correlations are definitely not physical models.* To understand it better, consider the example of statistics nonsense that was really done somewhere in northern Germany. (I don't give names here, for understandable reasons ;-)) It has been found that a correlation existed between the number of births of humans and the number of births of… (believe it or not) storks! So are we to say that it is a physical proven fact that… the storks bring the children? Think again about the answer. If you accept this, then certain human activities for children production will become unnecessary ;-)

The last thing to discuss is the principle of energy minimisation. Some of the transcendental stupids have blindly used it to defend their position "lower the energy of your brain and then the world will live in peace through transcendental meditation" - or similar bullshit. Apart from the fact that lowering the energy of our brains would kill us all (peace in its most unexpected form ;-)), they didn't even define what is the energy of the brain (or was it temperature? ;-)), they didn't consider that the fate of macroscopic phenomena is connected also to another quantity, the entropy, but instead of this they "guessed" a function with as many fitable parameters as possible, which they declared to the "state of the earth", and then fitted the guessed function to their (also undefined) transcendental meditation (mama mia!), and published this rubbish to the internet. Of course you won't find that shit on any serious scientific publication, oh no! Those "scientists" know that if they dared publish their garbage there, then… their time would come to take what is widely known as "transfer to lunar orbit". Even if the "guessed" function of undefined quantities were right (which it isn't), it wouldn't be a model and even less a theory, simply because it is (stupidly used) statistics and correlations of undefined quantities.

I must be in a very green state these days ;-) Anyway, another puzzle to think of: There is no recipe in this world that let's you construct any line segment with the length of exactly π, or of some expression that contains π in a transcendental manner. (Not the way JHM would like it be. The mathematical definition of transcendence is meant here ;-)) Nonetheless the electron, this tiny Mistviech, manages to "be in a distance" from the nucleus, whose mathematical expression contains $\frac{1}{\pi}$ as a factor!!! Though we can't produce such a distance with any mathematical construction in a finite number of steps, the electron can! Our mathematical description of nature has also its mysteries. And it is good that it does, or else it would be boring. But think about it for a moment. (And be sure to have a cup of very strong coffee somewhere near ;-))

The last example that we are going to examine using derivatives is the function that gives the concentration c_B of a substance B that appears in the reaction:

$$A \xrightarrow{k_1} B$$

$$B \xrightarrow{k_2} C$$

The above reaction mechanism means that substance A is converted to substance B with a rate k_1 and at the same time the substance B is converted to substance C with the a rate k_2. If we UNASSUME that mass is a continuum, and if the initial concentration of substance B is 0, then for the concentration c_B we have:

$$c_B = \frac{k_1\ c_{A_0}}{k_1 - k_2}\left(e^{-k_2 t} - e^{-k_1 t}\right)$$

where c_{A_0} is the initial concentration of substance A. The question is, does the concentration of substance B have an extremum at some certain time t? Let's see. We will find the roots of the first derivative with respect to t. Enter:

$$\frac{k1\ cA0}{k1-k2}\left(e^{-k2\ t}-e^{-k1\ t}\right)$$

and make a copy of the expression. Now enter t and press ∂ to get:

$$\frac{k1\ cA0}{k1-k2}\left(e^{-k2\ t}\cdot -k2-e^{-k1\ t}\cdot -k1\right)$$

Since all quantities that appear in the above expression are real an positive, switch to real mode, then enter the assumption list $\{cA0 \geq 0\quad k1 \geq 0\quad k2 \geq 0\quad t \geq 0\}$, press ASSUME, and drop the assumption list. Now enter t and press SOLVE. The HP49G will complain Not reducible to a rational expression. It is a shame that it doesn't solve this equation, but let's help it. Press ▶ and then DISTRIB to get:

$$e^{-k2\ t}\cdot -k2\,\frac{k1\ cA0}{k1-k2}-e^{-k1 t}\cdot -k1\,\frac{k1\ cA0}{k1-k2}$$

Press OBJ→, then twice ◀, and then = to get:

$$e^{-k2\ t}\cdot -k2\,\frac{k1\ cA0}{k1-k2}=e^{-k1 t}\cdot -k1\,\frac{k1\ cA0}{k1-k2}$$

Now press LN to get the natural logarithms of the left and the right hand sides. This operation takes some time to complete because the HP49G has to consider all the assumptions that we made. When it is ready you have:

$$LN\left(-\left(e^{-k1\ t}\cdot -k2\,\frac{k1\ cA0}{|k1-k2|}\right)\right)=LN\left(-\left(e^{-k1\ t}\cdot -k1\,\frac{k1\ cA0}{|k1-k2|}\right)\right)$$

Press TEXPAND. This operation also takes a bit more time. When it is ready you get:

$$-\left(LN\left(|k1-k2|\right)-\left(LN(cA0)+LN(k1)+LN(k2)-t\ k2\right)\right)=$$
$$-\left(LN\left(|k1-k2|\right)-\left(LN(cA0)+2\ LN(k1)-t\ k1\right)\right)$$

Press ▶ and then SOLVE to get:

$$t=\frac{LN(k1)-LN(k2)}{k1-k2}$$

Press LNCOLLECT to get:

$$t=\frac{LN\left(\frac{k1}{k2}\right)}{k1-k2}$$

This is the time at which the concentration c_B has its maximum. To find the expression for the maximum concentration, press DUP2, then SUBST. Now you have:

$$\frac{cA0}{k1-k2}\left(e^{-k2\left(\frac{LN\left(\frac{k1}{k2}\right)}{k1-k2}\right)}-e^{-k1\left(\frac{LN\left(\frac{k1}{k2}\right)}{k1-k2}\right)}\right)$$

If you now expand, then the HP49G will return a huge and unnecessarily complicated expression. If you try to take this expression in the EQW and apply EXP2POW separately to each exponential sub expression, then the HP49G will convert the exponentials to:

$\frac{1}{e^{k2\frac{LN\left(\frac{k1}{k2}\right)}{k1-k2}}}$ and $\frac{1}{e^{k1\frac{LN\left(\frac{k1}{k2}\right)}{k1-k2}}}$

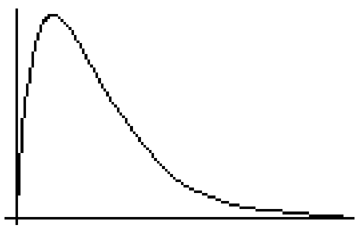
No simplification to:

$\left(\frac{k1}{k2}\right)^{\frac{k2}{k1-k2}}$ and $\left(\frac{k1}{k2}\right)^{\frac{k1}{k1-k2}}$

will be carried out, though under our assumptions the expression $\frac{k1}{k2}$ is positive and the expressions $\frac{k2}{k1-k2}$ and $\frac{k1}{k1-k2}$ are real. So you can only "imagine" that the expression for the maximum of c_B is:

$$\frac{cA0}{k1-k2}\left(\left(\frac{k2}{k1}\right)^{\frac{k2}{k1-k2}}-\left(\frac{k2}{k1}\right)^{\frac{k1}{k1-k2}}\right)$$

Anyway, if you give numeric values to the variables $ca0$, $k1$, and $k2$, and then plot the expression $\frac{cA0}{k1-k2}\left(e^{-k2\,t}-e^{-k1\,t}\right)$ with t as the independent variable, then you get a graph the shape of which is similar to the curve on the right.



Drop anything from the stack, until the expression $\frac{k1\ cA0}{k1-k2}\left(e^{-k2\,t}-e^{-k1\,t}\right)$ is on stack level 1. We are going to use it for making some thoughts about limits, removable and non-removable discontinuities. It is said quite often that for example the function $\frac{SIN(X)}{X}$ is not defined at $X=0$ because of division by 0, but things are not quite that simple. If it were that way, then the concentration $\frac{k1\ cA0}{k1-k2}\left(e^{-k2\,t}-e^{-k1\,t}\right)$ which is theoretically obtained by reaction kinetics and experimentally proven, would be… undefined in the case $k1=k2$, IP if substance A gets converted to substance B at exactly the same rate as substance B is converted to substance C. Which of course is absurd! We don't expect to have a chemical reaction which starts with a well defined concentration c_{A_0} and suddenly (through the influence of some holy ghost) the concentrations get undefined, do we? We have to look a little bit closer and realise that if $k1=k2$, then also $e^{-k2\,t}=e^{-k1\,t}$, which means that we have to work with limits. If the limit exists:

$$\lim_{k_1\to k_2}\frac{k1\ cA0}{k1-k2}\left(e^{-k2\,t}-e^{-k1\,t}\right)$$

then we can (or better, *we must*) accept that the function has to be replaced by this limit at $k1=k2$, in order to avoid… undefined concentrations. We have then a removable discontinuity. Enter $k1=k2$ and press lim. After some seconds the HP49G returns:

$$\frac{t\ k2\ cA0}{e^{t\,k2}}$$

which means that the concentration c_B is still defined and measurable as a function of the time t. If you differentiate and solve for t, you will get the solution:

$$t=\frac{1}{k2}$$

which is the time of the maximum concentration of substance B. The maximum concentration can then be found by substituting $t=\frac{1}{k2}$ in the expression $\frac{t\ k2\ cA0}{e^{t\,k2}}$, which gives the result $\frac{cA0}{e^{1}}$. Remove now all assumptions that we made by entering $\{cA0\ \ k1\ \ k2\ \ t\}$, pressing UNASSUME, and dropping the list from stack level 1. Since t belongs per default to the real variables of the CAS of the HP49G, enter t and press ADDTOREAL to add t to the real variables. Also, enter X and press STOVX to set variable VX to X, since the CAS has altered to $k1$ when we found the limit of c_B for $k1 = k2$.

Some criticism on the used model of this example. The expression that gives us the concentration of substance B at any time t is derived from the assumption that the reacting mass is a continuum. Which of course we know is wrong. Any material object consists of its molecules (discrete structure) and if the smallest unit that can react is one molecule, then the whole reaction will be probably also a non-continuous phenomenon. We can only have an integer number of molecules that react at some time. Of course, if we consider the huge number of molecules in an amount of a substance that we are able to weight, then the discrete behaviour can be indeed approximated by a continuous behaviour very well, since the smallest substance unit that can react - a molecule - is tiny in comparison to the mass of the substance. But if we want to be completely correct, we have to consider the possibility of a discontinuous model, and its consequences. We will do that later on, but now let's reconsider the problem that appears when $k1 = k2$, which we avoided by accepting that we have to use limits and to work with a removable discontinuity. The problem of concentrations of the used type of chemical reaction, can be formulated exactly and solved without any approximations. We will handle the formulation and solution of the problem when we deal with differential equations. For now it is enough to know that no mathematical approximations have to be used in order to derive $c_B = \frac{k_1\ c_{A_0}}{k_1 - k_2}\left(e^{-k_2\,t} - e^{-k_1\,t}\right)$. Why then does the solution behave this way? What brings the discontinuity, be it removable or not? Let's think about it. We have several possibilities to explain this behaviour. We could say that this is evidence for the impossibility to have two different chemical reactions that proceed with the same rate. This might sound not very reasonable, but if we take into consideration that two different chemical reactions involve different molecules and different reaction paths, and also the fact that the rates of the reactions are (in most cases) measured real quantities, we see that perhaps this is indeed what the discontinuity "wants to say to us". Measured quantities, especially real measured quantities, are always measured up to a certain degree of precision and accuracy. But because they are real, they will presumably be transcendent. (Not the way JHM wishes them to be ;-)) Such numbers are very hard to grasp exactly. Can two measured real reaction rates of different chemical reactions be exactly the same? Think of it. We will deal with such problems in another future marathon, but now let's continue on possible explanations about the reasons for the above discontinuity. One could also say that the fundaments of the used continuous model are not completely correct, and that a discrete model would avoid this problem. Acceptable critics, but we will see in a few minutes that this is not the reason for this behaviour.

Let's model the reaction using a discrete model for the case in which at the reaction start there are only molecules of the substance A. Some of these will react and be transformed to B. We denote the number of molecules of the substance A before the reaction with $A(n-1)$, and the number of molecules after the reaction with $A(n)$. The difference between the number of molecules before and after the reaction is exactly the number of molecules that reacted and got transformed to B. We assume that the number of molecules that react is proportional to the number of the existing molecules. This assumption is justified later on, when we see that experiments agree with the theory. That means:

$$A(n) - A(n-1) = -k1\ A(n-1) \Leftrightarrow A(n) = (1 - k1)\ A(n-1)$$

where $k1$ is the proportionality constant, which has to be less than 1 but positive. (Why?) Let's denote the initial number of molecules of substance A with $A0$. That is: $A(0) = A0$.

The above equations define a recursive sequence. If you have the Sequences, Series and Limits Marathon, you can enter:

$$\{A(n) = (1-k1)\ A(n-1)\quad \{A(0) = 0\}\}$$

and use the program $REC \rightarrow ANL$ to get the analytic closed form of this sequence:

$$\{A(n) = A0\ \left(-(k1-1)\right)^{n}\quad \{n\quad 0\quad +\infty\}\}$$

and then press **HEAD** to extract $A(n) = A0\ \left(-(k1-1)\right)^{n}$ out of the list. If you don't have the Sequences, Series and Limits Marathon, just enter the equation $A(n) = A0\ \left(-(k1-1)\right)^{n}$. As you can see, this is a decreasing geometric sequence, which shows that we at least caught the general behaviour of the reaction of substance A in our model. Since A gets transformed in B, and since there is no reaction that produces A, the number of molecules of substance A must decrease. Now let's consider substance B. According to the reaction, B is "created" by exactly those molecules of A, which have reacted. But it also reacts itself and gets transformed to C. The difference between the molecules before and each reaction "step" can be represented as:

$$B(n) - B(n-1) = k1\ A(n-1) - k2\ B(n-1)$$

which means that:

$$B(n) = k1\ A(n-1) + (1-k2)\ B(n-1)$$

The constant $K2$ plays the same role for B as $k1$ plays for A. If we now substitute what we have found for $A(n)$ in the above equation, we get:

$$B(n) = k1\ A0\ \left(-(k1-1)\right)^{n-1} + (1-k2)\ B(n-1)$$

We create a model for the case that no substance B is present at the start of the reaction, which means that $B(0) = 0$. These equations also define a recursion, which written in our notation is:

$$\{B(n) = k1\ A0\ \left(-(k1-1)\right)^{n-1} + (1-k2)\ B(n-1)\quad \{B(0) = 0\}\}$$

Unfortunately the program $REC \rightarrow ANL$ can't turn this recurrence to its analytic closed form. (Which shows how imperfect Nick's programming skills are ;-)) But using RSolve from Mathematica we get:

$$B(n) = \frac{k1\ A0\ \left((1-k2)^{n} - (1-k1)^{n}\right)}{k1-k2}$$

Did you notice something? Exactly the same problem , $k1-k2$ appears in the denominator. We still have the (removable) discontinuity!!! Again we must accept that when $k1 = k2$, we have to work with:

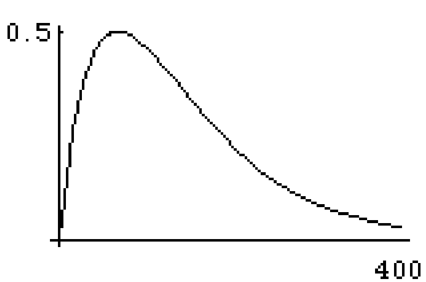
$$\lim_{k_1 \to k_2} \frac{k1\ A0\ \left((1-k2)^{n} - (1-k1)^{n}\right)}{k1-k2} = \frac{n\ k2\ A0\ (1-k2)^{n}}{1-k2}$$

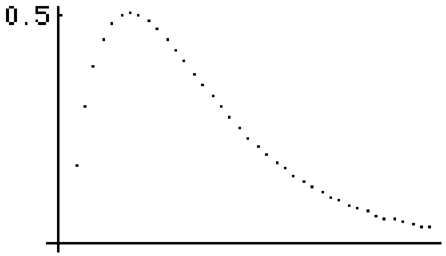
We see that it was not the assumption of mass continuum that was the reason for this problem. The discontinuity appears also when we use discrete modelling.

Notice also the similarity of the two expressions:

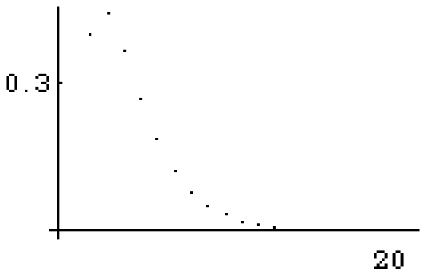
$$\frac{k1\ cA0}{k1-k2}\ \left(e^{-k2\ t} - e^{-k1\ t}\right) \text{ and } \frac{A0\ k1\ \left((1-k2)^{n} - (1-k1)^{n}\right)}{k1-k2}$$

If you give numeric values to the variables k1, k2, cA0 and A0, where cA0 = A0, and plot the first expression with t as independent variable and the second with n as independent variable, then you see that indeed both expressions describe the same thing. (Plot the second expression with the option _Connect unchecked.





The above plots were made with k1 = .02, k2 = .01, cA0 = A0 = 1. If you use much greater values for k1 and K2, you will notice that the discrete model produces a much higher maximum number of molecules for the concentration of B, and a much faster decrease in concentration of B. Can you explain why? (How clear can be a movie that shows a bullet flying at supersonic speeds, when each picture shot takes, say, 1 second?) Using the discrete model, we silently assume that for each n the molecules exist either in form of the substance A or in form of the substance B. But it is known that reactions like for example A → B follow a reaction path. Molecule A is not transformed immediately in molecule B, but it rather runs through many different phases, until it finally is converted to B. Our discrete model takes into consideration all molecules that either didn't even start reacting yet, or those which have completed their reaction. But it doesn't take into consideration those molecules which are reacting just now. How does this agree with the fact that small values for k1 and k2 give plots that agree with the continuous model, while bigger values give plots that are different from the continuous model? (Consider how big is the number of just reacting molecules compared with all the other molecules when k1 and k2 get greater.) How could we include the just reacting molecules in our discrete model? (The answer of this question is not easy and includes reaction probabilities.)



I hope that this marathon has been a source of puzzling twisted thoughts and of further ideas for heavy usage of the HP49G. We have seen that the machine has many "unexpected" features, which sometimes make pour lives really hard. But in general it is a real helper that can be used for purposes well beyond the level of education. (Which by no means implies that education level is low or easy.) In the next part we will continue with extrema of functions of more than one variables and similar examples taken from physics and chemistry.

Summer greetings - where is the fridge?
Nick.