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Differential Equations
Lesson 3
Bernoulli Equations

A Bernoulli Equation is one that can be put in the form

(1) $$\frac{dy}{dx} + v(x)y = w(x)y^a$$

where $a \neq 0,1$. Note that if a is zero the equation is linear and if it is one, the variables are separable. To solve (1), we define a new function, $u = y^{1-a}$. Solving this for y and its derivative gives us

(2) $$y = u^{\frac{1}{1-a}}$$

and

$$\frac{dy}{dx} = \frac{1}{1-a} u^{\frac{a}{1-a}} \frac{du}{dx}.$$

We substitute these into (1) to obtain

$$\frac{1}{1-a} u^{\frac{a}{1-a}} \frac{du}{dx} + v(x)u^{\frac{1}{1-a}} = w(x)u^{\frac{a}{1-a}},$$

which can be simplified to

(3) $$\frac{du}{dx} + (1-a)v(x)u = (1-a)w(x).$$

This is a linear equation that can be solved by the methods of Lesson 2 and has a solution of the form

$$u = CH(x) + P(x).$$

Applying (2), we now have that our solution to (1) is

$$y = [CH(x) + P(x)]^{\frac{1}{1-a}}.$$

Let us now consider the differential equation

(4) $$\frac{dy}{dx} - xy = 2xy^{-2}$$

as an example. This is a Bernoulli equation with a = -2, $v(x)$ = -x, and $w(x)$ = 2x. Using the substitution suggested in (2), we can write this in the form (3) as

$$\frac{du}{dx} - 3xu = 6x.$$

By display (2) of Lesson 2, this is a linear differential equation with $s(x)$ = -3x and $t(x)$ = 6x. Using the methods and terminology of Lesson 2 we have

$$H(x) = e^{-\int -3x dx} = e^{\frac{3}{2}x^2}$$

and

$$c(x) = \int \frac{6x}{e^{\frac{3}{2}x^2}} dx = -2e^{-\frac{3}{2}x^2} + C.$$

Then

$$u = \left(-2e^{-\frac{3}{2}x^2} + C\right) e^{\frac{3}{2}x^2} = Ce^{\frac{3}{2}x^2} - 2.$$

Then, by (2), the solution to (4) is

$$y = \left[Ce^{\frac{3}{2}x^2} - 2\right]^{1/3}.$$

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