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Differential Equations
Lesson 5
Systems of First Order Linear Equations

A system of n first order linear differential equations in n functions has the form

$$\begin{gathered} y_1'(x) = a_{11}y_1(x) + a_{12}y_2(x) + \cdots + a_{1n}y_n + E_1(x) \\ y_2'(x) = a_{21}y_1(x) + a_{22}y_2(x) + \cdots + a_{2n}y_n + E_2(x) \\ \vdots \\ y_n'(x) = a_{n1}y_1(x) + a_{n2}y_2(x) + \cdots + a_{nn}y_n + E_n(x). \end{gathered}$$

This system can be written in vector form as

$$\frac{d}{dx}\boldsymbol{y}(x) = \boldsymbol{A}\boldsymbol{y}(x) + \boldsymbol{E}(x)$$

where $\boldsymbol{y}(x) = \begin{bmatrix} y_1(x) \\ y_2(x) \\ \vdots \\ y_n(x) \end{bmatrix}$, $\boldsymbol{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ & \vdots & & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$, and $\boldsymbol{E}(x) = \begin{bmatrix} E_1(x) \\ E_2(x) \\ \vdots \\ E_n(x) \end{bmatrix}$. In general the a's can be functions, but we will restrict our consideration to constant coefficients. The system is **normal** on an interval $\boldsymbol{I}$ where $\boldsymbol{E}$ is continuous. The solution we will discuss will only be valid on an interval where the system is normal. If $\boldsymbol{E}$ is identically zero, the system is homogeneous. The general solution of such a system has the form

$$\boldsymbol{y}(x) = C_1\boldsymbol{H}_1(x) + C_2\boldsymbol{H}_2(x) + \cdots C_n\boldsymbol{H}_n(x) + \boldsymbol{P}(x)$$

where $\{\boldsymbol{H}_1, \boldsymbol{H}_2, \cdots, \boldsymbol{H}_n\}$ is an independent set of solutions to the homogeneous case and $\boldsymbol{P}$ is a particular solution to the non-homogeneous case. We will first consider only the homogenous case.

If α is an eigenvalue of $\boldsymbol{A}$ with corresponding eigenvector $\boldsymbol{v}$ then $e^{\alpha x}\boldsymbol{v}$ is a solution of the homogeneous system. The simplest case is when $\boldsymbol{A}$ has n distinct real eigenvalues. Consider the system

$$\begin{gathered} y_1'(x) = 2y_1(x) - 3y_2(x) \\ y_2'(x) = \ y_1(x) - 2y_2(x). \end{gathered}$$

The matrix $\boldsymbol{A} = \begin{bmatrix} 2 & -3 \\ 1 & -2 \end{bmatrix}$ has eigenvalues $\alpha = 1$ with corresponding eigenvector $\boldsymbol{v} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$ and $\alpha = -1$ with corresponding eigenvector $\boldsymbol{v} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. Thus the solutions is

$$\boldsymbol{y}(x) = C_1 e^x \begin{bmatrix} 3 \\ 1 \end{bmatrix} + C_2 e^{-x} \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

Converting this solution back to scalar form gives us

$$y_1(x) = 3C_1 e^x + C_2 e^{-e}$$
$$y_2(x) = C_1 e^x + C_2 e^{-e}.$$

To solve the above system on the calculator we first want to create a special directory for differential equations (See CTL 1). Let us call this directory DIFYQ. Now, within this directory we will create a custom menu (See CTL 23) with the following list of variables and commands; { A EGVL IDN AαI RREF DET }. If you now activate the custom menu the function keys should now be F1-A, F2-EGVL, F3-IDN, F4-AαI, F5-RREF, and F6-DET.

To use this custom menu to solve a system do the following:

1. Enter the matrix and store it in A.
2. Recall the matrix to the stack and press F2-EGVL to find the eigenvalues and write them down.

For each eigenvalue, α, do the following:

3. Recall A to the stack.
4. Put the eigenvalue on the stack
5. Put *n* (the dimension of the system) on the stack and press F3-IDN
6. Press × -. This creates the matrix$\boldsymbol{A} - \alpha \boldsymbol{I}$.
7. Store the result of step 6 in AαI. The need for this will become clear when we deal with more complex examples than the one above.
8. Recall AαI to the stack and press F5-RREF. This will give us what we need to find the related eigenvector(s) $\boldsymbol{v}$ associated with the eigenvalue α.
9. Write down the solution $H(x) = e^{\alpha x}\boldsymbol{v}$.

Let us go through these steps with the example above.

1. Use the matrix writer (See CTL 30) to put the matrix $\begin{bmatrix} 2 & -3 \\ 1 & -2 \end{bmatrix}$ on the stack then press LS F1-A.
2. Press F1-A then F2-EGVL. We see [1 -1] on the stack. This tells us the eigenvalues are 1 and -1. We write these down.

For the eigenvalue 1:

3. Press F1-A.
4. Press 1 ENTER.
5. Press 2 ENTER then F3-IDN
6. Press × -.
7. Press LS F4-AαI. This step is not really necessary for this simple example, but one should get in the habit of doing it since it may be useful for more complicated examples. The problem is that we will not know until after step 8 that we should have saved this result.
8. Press F4-AαI then F5-RREF. We see the matrix $\begin{bmatrix} 1 & -3 \\ 0 & 0 \end{bmatrix}$. This is telling us that the eigenvector $\boldsymbol{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$ has $v_1 - 3v_2 = 0$ or $v_1 = 3v_2$. We can choose

v_2 to be anything we wish except zero (recall that an eigenvector can never be the zero vector), but to make life as easy as possible for ourselves, we choose it to be 1. Thus the eigenvector associated with the eigenvalue 1 is $v = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$ and the solution associated with this eigenvalue is $H_1(x) = e^x \begin{bmatrix} 3 \\ 1 \end{bmatrix}$.

We repeat steps 3 through 8 with the eigenvalue -1 and find that $H_2(x) = e^{-x} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. Thus, we get the solution given above.

As another example let us consider the system

$$\begin{aligned} y_1'(x) &= -\ y_1(x) + \ y_2(x) \\ y_2'(x) &= -6y_1(x) + 4y_2(x) \\ y_3'(x) &= \qquad\quad + \ y_2(x) - y_3(x). \end{aligned}$$

1. Enter the matrix $\begin{bmatrix} -1 & 1 & 0 \\ -6 & 4 & 0 \\ 0 & 1 & -1 \end{bmatrix}$ and save it in F1-A
2. Press F1-A then F2-EGVL. We see [-1 2 1] on the stack, giving us 3 distinct eigenvalues.

After doing steps 3 through 8 with the eigenvalue -1 we have the matrix $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. (NOTE: in step 5 we must enter a 3 since we now have $n = 3$.) This is telling us that the corresponding eigenvector $\boldsymbol{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$ has $v_1 = v_2 = 0$ and v_3 can be anything we wish. To make life easy, we choose v_3 to be 1. Thus $H_1(x) = e^{-x} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ e^{-x} \end{bmatrix}$. Repeating steps 3 through 8 with the eigenvalue 1, we get the matrix $\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix}$. This is telling us that the corresponding eigenvector $\boldsymbol{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$ has $v_1 = v_3, v_2 = 2v_3$ and v_3 can be anything. Again, we choose v_3 to be 1. Then $H_2 = \begin{bmatrix} e^x \\ 2e^x \\ e^x \end{bmatrix}$. Finally, repeating steps 3 through 8 with the eigenvalue 2 gives us the matrix $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -3 \\ 0 & 0 & 0 \end{bmatrix}$. This is telling us that the corresponding eigenvector $\boldsymbol{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$ has $v_1 = v_3, v_2 = 3v_3$ and v_3 can be anything. We choose v_3 to be 1, then $H_3 = \begin{bmatrix} e^{2x} \\ 3e^{2x} \\ e^{2x} \end{bmatrix}$. We thus have the solution $\boldsymbol{y}(x) = C_1 \begin{bmatrix} 0 \\ 0 \\ e^{-x} \end{bmatrix} + C_2 \begin{bmatrix} e^x \\ 2e^x \\ e^x \end{bmatrix} + C_3 \begin{bmatrix} e^{2x} \\ 3e^{2x} \\ e^{2x} \end{bmatrix}$. Converting this back to scalar form gives us

$$\begin{aligned} y_1(x) &= \qquad\quad C_2 e^x + \ \ e^{2x} \\ y_2(x) &= \qquad\quad 2C_2 e^x + 3e^{2x} \\ y_3(x) &= C_1 e^{-x} + C_2 e^x + \ \ e^{2x}. \end{aligned}$$

There is a bit of a complication if some of the eigenvalues are complex. Because the elements of the matrix are all real the complex eigenvalues will occur in conjugate pairs. We only need one of the pair and its corresponding eigenvector to get two independent solutions. Let $\alpha + \beta i$ be an eigenvalue and let $\boldsymbol{v}$ be its corresponding eigenvector. Then

$$\boldsymbol{H}_1(x) = Re[e^{\alpha x}(\cos(\beta x) + i\sin(\beta x))\boldsymbol{v}]$$

and

$$\boldsymbol{H}_2(x) = Im[e^{\alpha x}(\cos(\beta x) + i\sin(\beta x))\boldsymbol{v}]$$

are two independent solutions. If we had chosen the other complex eigenvalue from this pair and its corresponding eigenvector, we would end up with the same solutions. (NOTE: to do the following example you will want to go into the CAS dialog box and check Approx and Complex and you will want flag 27 unchecked - see CTL 1). Let us consider the following example:

$$\begin{aligned} y_1'(x) &= -y_1(x) - y_2(x) \\ y_2'(x) &= 4y_1(x) - y_2(x). \end{aligned}$$

1. Enter the matrix $\boldsymbol{A} = \begin{bmatrix} -1 & -1 \\ 4 & -1 \end{bmatrix}$ and store it in F1-A
2. Recall $\boldsymbol{A}$ to the stack and press F2-EGNV. We see [(-1.,-2.) (-1.,2.)] on the stack. We will use the second eigenvalue, -1 + 2*i* to find the solutions, but it does not matter which one we choose.

After completing steps 3 through 8 we see the matrix $\begin{bmatrix} (1.,0.) & (0.,-.5) \\ (0.,0.) & (0.,0.) \end{bmatrix}$. This tells us that the eigenvector $\boldsymbol{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$ has $v_1 = .5iv_2$. To make our life easy, we choose $v_2 = 2$, so the corresponding eigenvector is $\boldsymbol{v} = \begin{bmatrix} i \\ 2 \end{bmatrix}$. Our two independent solutions, then, are

$$\boldsymbol{H}_1(x) = \mathrm{Re}\left(e^{-x}(\cos(2x) + i\sin(2x))\begin{bmatrix} i \\ 2 \end{bmatrix}\right) = \begin{bmatrix} -\mathrm{e}^{-x}\sin(2x) \\ 2e^{-x}\cos(2x) \end{bmatrix}$$

and

$$\boldsymbol{H}_2(x) = \operatorname{Im}\left(e^{-x}(\cos(2x) + i\sin(2x))\begin{bmatrix} i \\ 2 \end{bmatrix}\right) = \begin{bmatrix} e^{-x}\cos(2x) \\ 2e^{-x}\sin(2x) \end{bmatrix}.$$

Thus, the general solution is

$$\boldsymbol{y}(x) = C_1\begin{bmatrix} -\mathrm{e}^{-x}\sin(2x) \\ 2e^{-x}\cos(2x) \end{bmatrix} + C_2\begin{bmatrix} e^{-x}\cos(2x) \\ 2e^{-x}\sin(2x). \end{bmatrix}$$

Returning to scalar form, we now have the solutions

$$\begin{aligned} y_1(x) &= -C_1e^{-x}\sin(2x) + \ C_2e^{-x}\cos(2x) \\ y_2(x) &= \ 2C_1\,\mathrm{e}^{-x}\cos(2x) + 2C_2e^{-x}\sin(2x). \end{aligned}$$

Another potential source of complication is when some of the eigenvalues have multiplicity greater than 1. If we are lucky such eigenvalues will have a number of independent eigenvectors equal to the multiplicity, as in the example below:

$$\begin{aligned} y_1'(x) &= y_1(x) && + 4y_3(x) \\ y_2'(x) &= \quad 3y_2(x) && \\ y_3'(x) &= y_1(x) && + y_3(x). \end{aligned}$$

1. Enter the matrix $A = \begin{bmatrix} 1 & 0 & 4 \\ 0 & 3 & 0 \\ 1 & 0 & 1 \end{bmatrix}$ and save it in F1-A.
2. Press F1-A then F2-EGVL. We see that the eigenvalues are -1 and 3 with multiplicity 2.

After completing steps 3 through 8 with the eigenvalue 3 we see the matrix $\begin{bmatrix} 1 & 0 & -2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. This tells us that the eigenvector $\boldsymbol{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_2 \end{bmatrix}$ has $v_1 = 0v_2 + 2v_3$ with any choice we want for v_2 and v_3; thus we can get two independent eigenvectors by making different choices for v_2 and v_3. As usual, we will make choices that make our life as easy a possible. Our first pair of choices will be $v_2 = 1$ and $v_3 = 0$. This gives us the eigenvector $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ and the solution $\boldsymbol{H}_1(x) = \begin{bmatrix} 0 \\ e^{3x} \\ 0 \end{bmatrix}$. Our second choice will be $v_2 = 0$ and $v_3 = 1$. This will give us the eigenvector $\begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$ and the solution $\boldsymbol{H}_2(x) = \begin{bmatrix} 2e^{3x} \\ 0 \\ e^{3x} \end{bmatrix}$.

After completing steps 3 through 8 with the eigenvalue -1, we see the matrix $\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. This tells us that the eigenvector $\boldsymbol{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$ has $v_1 = -2v_3$ and $v_2 = 0$. We choose $v_3 = 1$, giving us the eigenvector $\begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}$ and solution $\boldsymbol{H}_3 = \begin{bmatrix} -2e^{-x} \\ 0 \\ e^{-x} \end{bmatrix}$. We now have the general solution

$$\boldsymbol{y}(x) = C_1 \begin{bmatrix} 0 \\ e^{3x} \\ 0 \end{bmatrix} + C_2 \begin{bmatrix} 2e^{3x} \\ 0 \\ e^{3x} \end{bmatrix} + C_3 \begin{bmatrix} -2e^{-x} \\ 0 \\ e^{-x} \end{bmatrix}.$$

Returning to scalar form, we have

$$\begin{aligned} y_1(x) &= \phantom{C_1 e^{3x}} \quad 2C_2 e^{3x} - 2C_3 e^{-x} \\ y_2(x) &= C_1 e^{3x} \\ y_3(x) &= \phantom{C_1 e^{3x}} \quad C_2 e^{3x} + C_3 e^{-x}. \end{aligned}$$

Unfortunately, we cannot count on always being lucky. In some cases an eigenvalue with multiplicity greater than one will have fewer eigenvectors than its multiplicity. In that case we can use the eigenvector(s) we do get to create some solution(s), but we must have a number of solutions equal to the multiplicity of the eigenvalue. To get the missing solutions we must resort to generalized eigenvectors. If $\boldsymbol{w}_0$ is a generalized eigenvector associated with an eigenvalue α then the solutions will have the form

(1) $$\boldsymbol{H}(x) = e^{\alpha x}\left(\boldsymbol{w}_0 + x\boldsymbol{w}_1 + \cdots + x^p \boldsymbol{w}_p\right)$$

where p is a positive integer equal to the number of **missing** eigenvectors and

(2) $$\boldsymbol{w}_k = \frac{1}{k}(\boldsymbol{A} - \alpha\boldsymbol{I})\boldsymbol{w}_{\boldsymbol{w-1}}, k = 1, 2, \cdots, p.$$

(Recall that A – $\boldsymbol{\alpha I}$ is what we save in step 7 of our algorithm; this is why we should save it as we could be using it repeatedly for this type of problem.) For example, suppose that we have an eigenvalue α with multiplicity 5 but only 2 independent eigenvectors. Then there are 3 missing eigenvectors, so we would have $p = 3$ in this case.

Let us consider the following example:

$$\begin{aligned} y_1'(x) &= -y_1(x) - y_2(x) \\ y_2'(x) &= y_1(x) - y_2(x) + y_3(x) \\ y_3'(x) &= + y_2(x) - y_3(x). \end{aligned}$$

After steps 1 and 2 we find that there is only one eigenvalue, -1, with multiplicity 3; and after steps 3 through 8 we find that the eigenvector $\boldsymbol{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$ has $v_1 = -v_3, v_2 = 0,$ and v_3 can be anything; thus we have only one independent eigenvector. We choose $v_3 = 1$, which leads to the solution $\boldsymbol{H}_1 = \begin{bmatrix} -e^{-x} \\ 0 \\ e^{-x} \end{bmatrix}$. Since the multiplicity of the eigenvalue is 3, and we have only one eigenvector, $p = 2$ in this case.

To find two generalized eigenvectors, we look for solutions to $(\boldsymbol{A} - \alpha\boldsymbol{I})^{p+1}\boldsymbol{w}_0 = \boldsymbol{0}.$ On the calculator we

10. Recall AαI to the stack.
11. Raise it to the p+1 power.
12. Press F5-RREF.

In this case we get the zero matrix, so we can pick any vectors we wish; in particular we can get three independent generalized eigenvectors. We only need two, but we must be sure to choose them so that the eigenvector we are using and the two generalized eigenvectors form an independent set. For our first choice, we will select $\boldsymbol{w}_0 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$. To apply definition (2),

13. Put $\boldsymbol{w}_0$ on the stack.
14. Repeat the following steps for k = 1, 2, …, p:
    a. Press F4-AαI
    b. Press RA
    c. Press ×.
    d. Put k on the stack
    e. Press ÷.
    f. Write down $\boldsymbol{w}_k$.
15. Write the solution as given in (1).

Note that when we do the above steps with our choice for $\boldsymbol{w}_0$ that $\boldsymbol{w}_2 = \begin{bmatrix} -.5 \\ 0 \\ .5 \end{bmatrix}$. I prefer to avoid fractions, so I went back and chose $\boldsymbol{w}_0 = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}$. Now the solution is

$$H_2(x) = e^{-x}\left(\begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix} + x\begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix} + x^2\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} (2 - x^2)e^{-x} \\ 2xe^{-x} \\ x^2e^{-x} \end{bmatrix}.$$

To find the third solution we choose $\boldsymbol{w}_0 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$. After completing steps 13 to 15 we find

$$\boldsymbol{H}_3(x) = e^{-x}\left(\begin{bmatrix}0\\1\\0\end{bmatrix} + x\begin{bmatrix}-1\\0\\1\end{bmatrix}\right) = \begin{bmatrix}-xe^{-x}\\e^{-x}\\xe^{-x}\end{bmatrix}.$$

The solution to the system, then is $\boldsymbol{y}(x) = C_1\begin{bmatrix}-e^{-x}\\0\\e^{-x}\end{bmatrix} + C_2\begin{bmatrix}(2-x^2)e^{-x}\\2xe^{-x}\\x^2e^{-x}\end{bmatrix} + C_3\begin{bmatrix}-xe^{-x}\\e^{-x}\\xe^{-x}\end{bmatrix}$.
Converting back to scalar form, we have

$$\begin{aligned}
y_1(x) &= [(-C_1 + 2C_2) - C_3x - C_2x^2]e^{-x}\\
y_2(x) &= [C_3 + 2C_2x]e^{-x}\\
y_3(x) &= [C_1 + C_3x + C_2x^2]e^{-x}.
\end{aligned}$$

In order to make sure that we have the complete general solution to the system, we need to know that the eigenvectors and generalized eigenvectors we use form an independent set. In this case it is quite clear that this condition is satisfied, but it may not be so obvious in more complex cases. To check this, form an $n \times n$ matrix from the n vectors used to generate the solution, with each column of the matrix one of the vectors. If the determinant of this matrix is not zero, the vectors are independent. For this problem the matrix would be $\begin{bmatrix}-1 & 1 & 0\\0 & 0 & 1\\1 & 0 & 0\end{bmatrix}$. Put this matrix on the stack and press F6-DET. We see the determinant is 1, not zero, so we do have the general solution to the system.

Finally, we must consider the non-homogeneous case. We use the method of undetermined coefficients, similar to what we did for the non-homogeneous n-th order equations in Lesson 4. We first solve the related homogeneous case, then assume the C's are functions c(x) to find $\boldsymbol{P}$(x). That is, we assume

$$\textbf{(3)} \qquad \boldsymbol{P}(x) = c_1(x)\boldsymbol{H}_1(x) + c_2(x)\boldsymbol{H}_2(x) + \cdots + c_n(x)\boldsymbol{H}_n(x)$$

where the $\boldsymbol{H}$'s are the solutions to the related homogeneous case. To find the c's we must solve the system

$$\begin{gathered}
h_{11}(x)c_1'(x) + h_{12}(x)c_2'(x) + \cdots + h_{1n}(x)c_n'(x) = E_1(x)\\
h_{21}(x)c_1'(x) + h_{22}(x)c_2'(x) + \cdots + h_{2n}(x)c_n'(x) = E_2(x)\\
\vdots\\
h_{n1}(x)c_1'(x) + h_{n2}(x)c_2'(x) + \cdots + h_{nn}(x)c_n'(x) = E_n(x)
\end{gathered}$$

for each c'(x) where $h_{jk}(x)$ is the j$^{\text{th}}$ component of $\boldsymbol{H}_{\text{k}}$. We then integrate each c'(x) to find the c's. Let us consider the following example:

$$\begin{aligned}
y_1'(x) &= y_1(x) + \phantom{3y_2(x) +{}} 4y_{3(x)} - 2e^x\\
y_2'(x) &= \phantom{y_1(x) +{}} 3y_2(x) + 9x\\
y_3'(x) &= y_1(x) + \phantom{3y_2(x) +{}} y_3(x) + e^x.
\end{aligned}$$

After following steps 1 through 9 we find the solution to the homogeneous case is

$$\boldsymbol{y}(x) = C_1\begin{bmatrix} -2e^{-x} \\ 0 \\ e^{-x} \end{bmatrix} + C_2\begin{bmatrix} 0 \\ e^{3x} \\ 0 \end{bmatrix} + C_3\begin{bmatrix} 2e^{3x} \\ 0 \\ e^{3x} \end{bmatrix}.$$

The system we must solve to find the derivatives of the c's is

$$\begin{aligned} -2e^{-x}c_1'(x) + \phantom{e^{3x}c_2'(x)} \quad 2e^{3x}c_3'(x) &= -2e^{x} \\ e^{3x}c_2'(x) \phantom{2e^{3x}c_3'(x)} &= 9x \\ e^{-x}c_1'(x) + \phantom{e^{3x}c_2'(x)} \quad e^{3x}c_3'(x) &= e^{x}. \end{aligned}$$

This system is easier to solve by hand than on the calculator. The solutions are

$$\begin{aligned} c_1'(x) &= e^{2x} \\ c_2'(x) &= 9xe^{-3x} \\ c_3'(x) &= 0. \end{aligned}$$

For the sake of completeness, we discuss how it can be solved on the calculator. Enter the augmented matrix for the system, $\begin{bmatrix} -2e^{-x} & 0 & 2e^{3x} & -2e^{x} \\ 0 & e^{3x} & 0 & 9x \\ e^{-x} & 0 & e^{3x} & e^{x} \end{bmatrix}$, into the calculator. Press F5-RREF. Press DA to put the matrix into the matrix writer. The solutions are in the forth column. It is easy to see that the solution for $c_3'(x) = 0$, but the other two are quite messy looking. To make them look a little nicer press CANCEL and ENTER, to get back to the main screen and put a second copy of the matrix on the stack. Put {1 4} on the stack, type the command GET, and press ENTER, then EVAL. This will extract the element from row 1, column 4 from the matrix and simplify it. The resulting fraction is still not reduced, but it clearly will reduce to e^{2x}. The GET command deletes the matrix from level 2 of the stack, which is why we needed to put a second copy of the matrix on the stack. Drop this solution then repeat the steps above with {2 4} to get the next solution. Now integrate (See CTL 17) to get

$$\begin{aligned} c_1(x) &= \frac{1}{2}e^{2x} \\ c_2(x) &= -(3x+1)e^{-3x} \\ c_3(x) &= 0. \end{aligned}$$

We now combine these results and the solutions to the homogeneous case as indicated in our assumption (3) to get a particular solution to the non-homogeneous case:

$$\boldsymbol{P}(x) = \frac{1}{2}e^{2x}\begin{bmatrix} -2e^{-x} \\ 0 \\ e^{-x} \end{bmatrix} - (3x+1)e^{-3x} + 0\begin{bmatrix} 2e^{3x} \\ 0 \\ e^{3x} \end{bmatrix}.$$

Combining this with the solution to the homogeneous case, we have the general solution to our system in scalar form:

$$
\begin{aligned}
y_1(x) &= -2C_1e^{-x} + 2C_3e^{2x} - e^x \\
y_2(x) &= C_2e^{3x} - 3x - 1 \\
y_3(x) &= C_1e^{-x} + C_3e^{3x}.
\end{aligned}
$$

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