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Differential Equations
Lesson 1
Separable Variables

In this lesson we will consider some simple cases of first order differential equations. These are equations that can be written in the form

(1)
$$\frac{dy}{dx} = f(x,y).$$

These equations fall into several classes. The simplest class is if y is absent from the right hand side of the equation. That is, the equation takes the form

$$\frac{dy}{dx} = f(x).$$

The solution is then

$$y = \int f(x)dx + C.$$

This can be solved using the method for finding anti-derivatives from CTL17. Remember that the calculator must be in exact mode (See CTL 1) for integration and differentiation. For example consider the differential equation

$$\frac{dx}{dy} = x^2e^x.$$

Using the equation writer (CTL 2) put x^2e^x on the stack and press LS > CALC > F6-INTVX to see the solution $(x^2 - 2x + 2)e^x$. We must, of course, add the arbitrary constant, so the solution is

(2)
$$y = (x^2 - 2x + 2)e^x + C.$$

If we have been given an initial condition such as $y(0) = 5$ we can now solve for C. Substituting $y = 5$ and $x = 0$ into (2) leads to $C = 3$. If the expression $(x^2 - 2x + 2)e^x$ is still on level 1 of the stack, the solution can be defined as a function on the calculator as follows: press DA > F1-EDIT, add Y(X)= to the left of the expression and +3 to the right end, then press ENTER > ENTER > LS > DEF (CTL 2).

The next case of (1) we will consider is when the variables are separable. In this case the function $f(x,y)$ can be written as shown below:

(3)
$$\frac{dy}{dx} = f(x,y) = g(x)h(y).$$

This can be rewritten in terms of differentials as

$$\frac{dy}{h(y)} = g(x)dx.$$

As long as $h(y) \neq 0$, we can use the methods from CTL 17 to integrate both side to find functions H and G such that

$$H(y) = G(x) + C.$$

If possible, we now solve for y to get a solution of the form $y = F(x)$.

Our solution was based on the assumption that $h(y) \neq 0$. Suppose that there is a real number r such that $h(r) = 0$. Then, clearly, the constant function $y = r$ is a solution to (3).

We will consider two examples. First, let us solve the differential equation

$$\frac{dy}{dx} = y^2. \tag{4}$$

Writing this in terms of differentials gives us $y^{-2}dy = dx$, and integrating both sides gives us $-y^{-1} = x + C$. Solving this for y gives us the solution

$$y = \frac{-1}{x + C}.$$

Note, however, that we are not done. This solution was based on the assumption that $y \neq 0$, and clearly the constant function $y = 0$ also satisfies (4). The solution to (4), therefore, should be written as $y = \frac{-1}{x+C}$ or $y = 0$.

As our second example consider the differential equation

$$\frac{dy}{dx} = \frac{yx - x}{x^2 + 1}. \tag{5}$$

Writing this in differential form gives us

$$\frac{dy}{y - 1} = \frac{x}{x^2 + 1}.$$

Using the method from CTL 17 and the assumption $y \neq 1$ gives us

$$\ln(|y - 1|) = \frac{1}{2}\ln(|x^2 + 1|) + C_1.$$

Notice that since $x^2 + 1$ is always positive, the absolute value symbol on the right is not necessary, so we will omit it. Using the properties of logarithms and exponentiating both sides gives us

$$|y - 1| = e^{C_1}\sqrt{x^2 + 1} = C\sqrt{x^2 + 1}, \qquad C > 0.$$

Note that since e^{C_1} is a positive constant we have replaced it with $C > 0$. Solving this for y gives us

$$y = 1 \pm C\sqrt{x^2 + 1}, \qquad C > 0.$$

We now observe that we can replace $\pm C$ with $+ C$ if we allow C to be positive or negative. We also observe that the constant function $y = 1$ is also a solution to (5), and that solution is obtained by allowing C to be zero. Our general solution, then, is

$$y = 1 + C\sqrt{x^2 + 1}$$

where C can be any real number.

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