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Differential Equations
Lesson 4
N[th] Order Linear Equations

In general an n[th] order linear differential equation has the form

(1) $$a_n(x)\frac{d^ny}{dx^n}+a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}}+\cdots+a_1(x)\frac{dy}{dx}+a_0(x)y=E(x).$$

The equation is **normal** on an interval **I** if all the a_i's and *E* are continuous on **I** and a_n is never zero for *x* in **I**. It is homogeneous if *E* is the zero function. The general solution of (1) has the form

(S) $$y=C_1H_1(x)+C_2H_2(x)+\cdots+C_nH_n(x)+P(x)$$

where $\{H_1,H_2,\ldots,H_n\}$ is an independent set of solutions to the homogeneous case, *P* is a particular solution to the non-homogeneous case, and the *C*'s are arbitrary constants. Recall that a set of functions $\{H_1,H_2,\ldots,H_n\}$ is independent if $C_1H_1(x)+C_2H_2(x)+\cdots+C_nH_n(x)=0$ implies that all of the *C*s are zero.

In this lesson we will restrict our attention to the case with all of the a's constant and will start with the assumption that the equation is homogeneous. Thus, we are looking at an equation of the form

(2) $$a_n\frac{d^ny}{dx^n}+a_{n-1}\frac{d^{n-1}y}{dx^{n-1}}+\cdots+a_1\frac{dy}{dx}+a_0y=0$$

with $a_n\neq 0$. Associated with (2) is the polynomial, called the characteristic polynomial,

$$p(t)=a_nt^n+a_{n-1}t^{n-1}+\cdots+a_1t+a_0.$$

The simplest case is when the characteristic polynomial has *n* distinct real zeros, $\{\alpha_1,\alpha_2,\ldots,\alpha_n\}$. The independent solutions to (2) then have the form $H_i(x)=e^{\alpha_ix}, i=1,2,\ldots,n$. As an example consider the 4[th] order differential equation

(3) $$2\frac{d^4y}{dx^4}-9\frac{d^3y}{dx^3}+3\frac{d^2y}{dx^2}+14\frac{dy}{dx}=0.$$

The related characteristic polynomial is $p(t)=2t^4-9t^3+3t^2+14t$. Using the polynomial solver (see CTL 32) we find that the solution set to the characteristic polynomial is {-1, 0, 2, 7/2}. Thus, the general solution of (3) is

(4) $$y(x)=C_1e^{-x}+C_2+C_3e^{2x}+C_4e^{\frac{7}{2}x}.$$

Now suppose we have been given the initial conditions $y(0)=0, y'(0)=1, y''(0)=5$

$y'''(0) = 7$. Letting $x = 0$ in (4) and its first three derivatives gives us the following system of four equations in four unknowns:

$$C_1 + C_2 + C_3 + C_4 = 0$$
$$-C_1 + 0 + 2C_3 + \frac{7}{2}C_4 = 1$$
$$C_1 + 0 + 4C_3 + \frac{49}{4}C_4 = 5$$
$$-C_1 + 0 + 8C_3 + \frac{343}{8}C_4 = 7.$$

With the calculator in exact mode (see CTL 1) use the Matrix Writer (see CTL 30) and the RREF command (see CTL 31) we find that the solution to this system is $C_1 = 1$, $C_2 = -2$, $C_3 = 1$, and $C_4 = 0$. Thus the solution of (2) with the given initial conditions is $y(x) = e^{-x} - 2 + e^{2x}$.

Now suppose that the characteristic polynomial has some complex zeros. Since the coefficients are all real, the complex zeros will occur in conjugate pairs. We will again assume that the complex zeros are unique. In that case, the complex pair $\alpha \pm \beta i$ leads to the two independent solutions $H_1(x) = e^{\alpha x}\cos(\beta x)$ and $H_2 = e^{\alpha z}\sin(\beta x)$. As an example consider the differential equation

$$\frac{d^5y}{dx^5} - 9\frac{d^4y}{dx^4} + 29\frac{d^3y}{dx^3} - 61\frac{d^2y}{dx^2} + 100\frac{dy}{dx} - 100y = 0.$$

Using the polynomial solver (see CTL 32) we find that the solution set of the characteristic polynomial for this equation is $\{5, 2i, -2i, 2 + i, 2 - i\}$. Thus the general solution of the differential equation is $y(x) = C_1e^{5x} + C_2\cos(2x) + C_3\sin(2x) + C_4e^{2x}\cos(x) + C_5e^{2x}\sin(x)$.

The final complication occurs if a zero of the characteristic polynomial has multiplicity $m > 1$. In that case if $H(x)$ is the solution related to such a zero, then $xH, x^2H, \ldots, x^{m-1}H$ are also independent solutions. As an example let us consider the differential equation

$$\frac{d^7y}{dx^7} - 6\frac{d^6y}{dx^6} + 14\frac{d^5y}{dx^5} - 20\frac{d^4y}{dx^4} + 25\frac{d^3y}{dx^3} - 22\frac{d^2y}{dx^2} + 12\frac{dy}{dx} - 8 = 0$$

Using the factoring method from CTL 32 we see that the zeros are $t = 2$ with multiplicity $m = 3$ and $t = 0 \pm i$ with multiplicity $m = 2$. Thus the solution to our differential equation is

$$y(x) = C_1e^{2x} + C_2xe^{2x} + C_3x^2e^{2x} + C_4\cos x + C_5\sin x + C_6x\cos x + C_7x\sin x.$$

We now consider how to deal with (1) if it is not homogeneous. We will use a method called Variation of Parameters. We are still assuming that the a's are all constants and that a_n is not zero, so we can divide by it. This gives us an equation in the form

$$\frac{d^ny}{dx^n} + b_{n-1}\frac{d^{n-1}y}{dx^{n-1}} + \cdots + b_1\frac{dy}{dx} + b_0y = F(x). \tag{5}$$

Suppose the solution to the homogeneous case of (5) is $y(x) = C_1H_1(x) + \cdots + C_nH_n(x)$. We make the assumption that the solution to the non-homogeneous case has the same form except that the C's are functions of x, not constants. We will use lower case c's for these functions to distinguish them from the constants. To find the derivatives of these functions we solve the following system of linear equations

$$\begin{gathered}
c_1'(x)H_1(x) + c'_2(x)H_2(x) + \cdots + c'_n(x)H_n(x) = 0 \\
c_1'(x)H'_1(x) + c'_2(x)H'_2(x) + \cdots + c'_n(x)H'_n(x) = 0 \\
\vdots \\
c_1'(x)H_1^{(n-2)}(x) + c'_2(x)H_2^{(n-2)}(x) + \cdots + c'_n(x)H_n^{(n-2)}(x) = 0 \\
c_1'(x)H_1^{(n-1)}(x) + c'_2(x)H_2^{(n-1)}(x) + \cdots + c'_n(x)H_n^{(n-1)}(x) = F(x).
\end{gathered}$$

then integrate each c_i' to find c_i.

As an example let us consider the differential equation

$$2\frac{d^3y}{dx^3} - 12\frac{d^2y}{dx^2} + 24\frac{dy}{dx} - 16y = \frac{e^{2x}}{x}.$$

We will add the stipulation that x is positive. Since the leading coefficient is not 1, we divide by it, giving us

$$\frac{d^3y}{dx^3} - 6\frac{d^2y}{dx^2} + 12\frac{dy}{dx} - 8y = \frac{e^{2x}}{2x}.$$

We assume the previous step can be done in our heads, but the rest of the steps will use the calculator. To make sure that yours behaves the same as indicated below, press MODE > F3-CAS and make sure that both _Numeric and _Approx are unchecked, putting the calculator in exact mode. (See CTL 1). The characteristic polynomial for this differential equation is $t^3 - 6t^2 + 12t - 8$. We will use the factoring method to find the zeros of this polynomial. (See CTL 32.) Put the polynomial on the stack and press RS ALG F3-FACTO. We see that the zero is 2 with multiplicity 3, hence, the three independent solutions to the homogeneous case of our differential equation are $H_1(x) = e^{2x}$, $H_2(x) = xe^{2x}$, and $H_3(x) = x^2e^{2x}$. The next step is to solve the system of equations

$$\begin{gathered}
c_1'(x)H_1(x) + c_2'(x)H_2(x) + c_3'H_3(x) = 0 \\
c_1'(x)H_1'(x) + c_2'(x)H_2'(x) + c_3'(x)H_3'(x) = 0 \\
c_1'(x)H_1''(x) + c_2'(x)H_2''(x) + c_3'(x)H_3''(x) = \frac{e^{2x}}{2x}
\end{gathered}$$

for the derivatives of the c's. We will be using methods from CTL 8, CTL 17, CTL 31, and CTL 33 to do this.

Place e^{2x} on the stack then perform the following sequence of steps:

LS CALC ENTER F5-DERVX EVAL ENTER F5-DERVX EVAL
LS PRG F5-TYPE 3 F2-→ARRY

This has created a vector with H_1 and its two derivatives. Now place xe^{2x} on the stack and repeat the above to create a vector with H_2 and its derivatives, then repeat with x^2e^{2x} to put H_3 and its derivatives on the stack. Now create the vector $\begin{bmatrix} 0 & 0 & \frac{e^{2x}}{2x} \end{bmatrix}$ on the stack. Now press LS MTH F2-MATRX F4-COL 4 F2-COL→ to create the augmented matrix for the system we need to solve. Now F6-MATRX F3-FACTOR F1-RREF gives us the reduced matrix with the solutions in the last column. Note that the solutions are not reduced; each has e^{2x} in both the numerator and denominator. To simplify these, press NXT F6-MATRX F4-COL F1-→COL. Press the back arrow to delete the 4 that is on level 1: of the stack, It is not necessary, but you may do SWAP DROP three times to get rid of the three columns of the matrix we no longer need. Now press LS PRG F5-TYPE F1-OBJ→ and the back arrow to delete the 3 from level 1: of the stack. The value of c_3' is now on level 1: of the stack. To reduce the fraction, press EVAL, then LS CALC F6-INTVX to find $c_3(x) = \frac{1}{4}\ln(|x|) + C3$. Remember that you must provide the arbitrary constant when using the calculator to find an indefinite integral, and rather that a subscript, we will us $C3$ on the calculator. Since we are assuming that x is positive, you may wish to edit this to remove the absolute value function. Now put x^2e^{2x} on the stack and multiply.

Press TOOL F3-STACK F6-UNROT. (See CTL 33) This puts the value of c_2' on level 1: of the stack. EVAL will reduce the fraction and LS CALC F6-INTVX will give us the solution $c_2(x) = -\frac{x}{2} + C2$ after we add the arbitrary constant. Put xe^{2x} on the stack and multiply.

Press TOOL F3-STACK F6-UNROT to put the value of c_1' on level 1: of the stack. Now EVAL will reduce the fraction and LS CALC F6-INTVX will perform the integration. In this case you should press EVAL to simplify the answer. After providing the arbitrary constant we have $c_1(x) = \frac{x^2}{8} + C1$. Place e^{2x} on the stack and multiply. Now press TOOL F3-STACK F6-UNROT then press the plus sign twice. If we now press TOOL F2-VIEW (See CTL 33) we can see the solution

$$\left(\frac{x^2}{8} + C1\right)e^{2x} + \left(\frac{-x}{2} + C2\right)xe^{2x} + \left(\frac{1}{4}\ln(x) + C3\right)x^2e^{2x}.$$

Note that this can be rewritten as

$$C1e^{2x} + C2xe^{2x} + \left(C3 + \frac{1}{8} - \frac{1}{2}\right)x^2e^{2x} + \frac{1}{4}\ln(x)\,x^2e^{2x}.$$

If we now collapse the 1/8 and -1/2 into $C3$, and convert back to subscripts, we can write this as

$$y = C_1e^{2x} + C_2xe^{2x} + C_3x^2e^{2x} + \frac{1}{4}\ln(x)\,x^2e^{2x},$$

which is of the form (S) above.

If you would like to check that $\frac{1}{4}\ln(x)\,x^2e^{2x}$ is in fact a particular solution to the non-homogeneous case, put it on the stack and do the following

ENTER 16 +/- × RA LS CALC F5-DERVX ENTER 24 × RA
F5-DERVX ENTER 12 +/- × RA F5-DERVX 2 × + + + EVAL

You should now see $\frac{e^{2x}}{x}$ on the stack.

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