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Differential Equations
Lesson 2
First Order Linear Equations

A first order linear differential equation is one that has the form

$$a(x)\frac{dy}{dx} + b(x)y = E(x) \tag{1}$$

where $a(x)$, $b(x)$, and $E(x)$ are functions. If E is the constant zero function, (1) is said to be ***homogeneous***. If there is an open interval **I** such that a, b, and E are continuous on **I** and $a(x)$ is never zero for x in **I**, then (1) is said to be ***normal*** on **I**. If (1) is normal on some interval **I** we can divide by $a(x)$ and write the equation in the form

$$\left\{\frac{dy}{dx} + \frac{b(x)}{a(x)}y = \frac{E(x)}{a(x)}\right\} = \left\{\frac{dy}{dx} + s(x)y = t(x)\right\}. \tag{2}$$

We note that if the equation is homogeneous the variables are separable and the equation can be solved by the methods of Lesson 1. The solution to the homogeneous case is then

$$y = Ce^{-\int s(x)dx} = CH(x) \tag{3}$$

where C is an arbitrary constant. We now make the bold assumption that the solution to the non-homogeneous case has the same form except that the constant C should be replaced by a function $c(x)$. If $c(x)H(x)$ is to be a solution, it must satisfy (2), that is, we must have

$$\frac{d[c(x)H(x)]}{dx} + s(x)c(x)H(x) = t(x),$$

$$\frac{dc(x)}{dx}H(x) + c(x)\frac{dH(x)}{dx} + s(x)c(x)H(x) = t(x),$$

$$\frac{dc(x)}{dx}H(x) + c(x)\left[\frac{dH(x)}{dx} + s(x)H(x)\right] = t(x).$$

We note that since $H(x)$ is a solution to the homogeneous case, the expression in the brackets above is zero. Thus, we can remove that term and solve for

$$\frac{dc(x)}{dx} = \frac{t(x)}{H(x)}$$

and

(4) $$c(x) = \int \frac{t(x)}{H(x)} dx + C.$$

We now substitute this function for *C* in (3) and obtain the general solution of (1)

(5) $$y = (c(x) + C)H(x) = CH(x) + c(x)H(x) = CH(x) + P(x)$$

where, *C* is an arbitrary constant, *H*(*x*) is a solution to the homogeneous case and *P*(*x*) is a particular solution to the non-homogeneous case. Remember, however, that this solution is only valid over the interval **I** over which (1) is normal.

As an example, we will consider the differential equation

(6) $$(x^2 - 1)\frac{dy}{dx} + y = \sqrt{x-1}.$$

This equation is normal over the open interval $x > 1$, so we will make that restriction. The equation can then be written in the form of (2)

$$\frac{dy}{dx} + \frac{y}{x^2-1} = \frac{\sqrt{x-1}}{x^2-1}$$

Where $s(x) = \frac{1}{x^2-1}$ and $t(x) = \frac{\sqrt{x-1}}{x^2-1}$. As indicated in (3), and using the methods of CTL 17 we now find

(7) $$H(x) = e^{-\left(-\frac{1}{2}\ln(x+1)+\frac{1}{2}\ln(x-1)\right)} = \sqrt{\frac{x+1}{x-1}}.$$

We now find that

$$\frac{t(x)}{H(x)} = \frac{\frac{\sqrt{x-1}}{x^2-1}}{\sqrt{\frac{x+1}{x-1}}} = \frac{1}{(x+1)^{3/2}}.$$

Now, applying (4) and the methods of CTL 17 we find that

$$c(x) = \frac{-2}{\sqrt{x+1}} + C.$$

Finally, combining this with the result (7) as indicated in (5) we have the solution to (6) for $x > 1$

$$y = C\sqrt{\frac{x+1}{x-1}} + \frac{-2}{\sqrt{x-1}},$$

which is in the form of (5) with H as shown in (7) and $P(x) = \frac{-2}{\sqrt{x-1}}$.

We leave it to the reader to verify that this is in fact the correct solution.

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