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INSTRUCTIONS FOR USING THE CUBIC SPLINE SYSTEM

In the directory SPLINE we find five active function keys; F1-INPT, F2-S, F3- USR , F4-ACTM, and F5-FUNC. INPT is the program you will use to enter new data or to choose a previously saved data set. Once the data has been entered or selected, S can be used repeatedly to approximate the function at as many points as desired. USR is a directory that contains all of the saved files. The only time you will need to enter that directory is if you have a old data set you are sure you will never need again and wish to purge. ACTM (ACTive Matrix) is a matrix that contains the information necessary for the function currently being approximated. If you turn the calculator off then return to continue approximating the same function you were using previously, the information is still in ACTM, so it is not necessary to re-run INPT. That is only necessary when you want to switch to a different data set. ACTM has a zero in it before the spline programs are used for the first time. FUNC contains all of the procedures eight of them) that actually do all of the work. INPT and S simply call on the procedures in FUNC. Once your system is operating properly you will never need to go into FUNC.

Once you have loaded the spline system into your calculator, try the following to (a) make sure it is working properly and (b) to learn how to use the system. Make sure the calculator is in Approximate mode and set the display to Fix 3 for these examples. As our first example we will approximate the sine function from 0 to π/2 using 7 nodes with a free spline.

Get into the Spline directory. Press F1-INPT. Press DA to select "Create free spline" and press F6-OK. We must enter a name for the file we create. Let us call it FSIN for Free SINe. Notice that the α lock has been set, so it is not necessary to press AS, simply type FSIN and press ENTER. We must now enter the number of data points we will use, 7 in this case. Type 7 and press ENTER. We are now ready to enter the data in the Table 1 below.

x in radian	Y = sin(x)
0π = 0.000	0.000
π/12 = 0.262	0.259
π/6 = 0.524	0.500
π/4 = 0.785	0.707
π/3 = 1.047	0.866
5π/12 = 1.309	0.966
π/2 = 1.571	1.000

Table 1

Type 0 in the x field, press F6-OK, type 0 in the y field, and press F6-OK. When you are sure both values are correct press F6-OK to accept them. The calculator now goes on to the next set. Notice that the prompt at the top of the screen tells you which set it is expecting. Continue entering the remaining 6 sets from the table above. When the last set has been entered the calculator will take some time to calculate the coefficients for the spline. The time required is proportional to the number of data points. If you are curious to see the coefficients, press F4-ATCM. You will see a 5x7 (in this case) matrix with the information the calculator needs to approximate the sine for values between 0 and π/2. As a general rule the maximum error in an approximation will occur at the midpoint between consecutive nodes, so let us approximate the sine at the six midpoints between our nodes. The first one is π/24 = 0.131. Place 0.131 in the command line or on the stack and press F2-S. Repeat for the other midpoints. The results are shown in the table below.

x in radians	Approximation of sin(x)	Actual sin(x)
$\pi/24 = 0.131$	0.131	0.131
$\pi/8 = 0.393$	0.383	0.383
$5\pi/24 = 0.654$	0.608	0.608
$7\pi/24 = 0.916$	0.793	0.793
$3\pi/8 = 1.178$	0.925	0.924
$11\pi/24 = 1.440$	0.988	0.991

If your approximations do not agree with those above, you need to check your input data and/or your programs. You can check that you entered the correct data by looking at the top two rows of the matrix ACTM. The top row should contain the *x* values you entered and the second row should contain the *y* values you entered. If a whole row is off, it is probably the result of an error in your program. If only a few values are off, it is probably the result of an error in data entry.

The numbers in the second row (the input *y* values) are the constant terms in the polynomials that make up the spline. The number under x_i is the constant term of the cubic polynomial that spans the interval $[x_i, x_{i+1}]$. The number below it in the third row is the coefficient of $(x - x_i)$, the number below that in the fourth row is the coefficient of $(x - x_i)^2$, and the number below that in the fifth row is the coefficient of $(x - x_i)^3$.

Notice that the approximations are exactly the same as the correct value for the first few entries. This is deceptive accuracy that is due to an accident of the function we chose to approximate. Recall that the free (or natural) spline assumes the second derivative of the function being approximated is zero at the end point. In this case, the second derivative of the sine function is zero at the left end, leading to exceptionally accurate approximations at that end of the interval. At the right end, however, the second derivative of the sine junction is -1, not zero, so the approximations are not quite so accurate at that end.

Now let us try approximating the sine with a clamped spline. Press F1-INPT, but this time choose "Create clamped spline" and press F6-OK. Enter CSINE for the name of the file and press ENTER. The next thing is to enter the derivative values at the two ends. The derivative of the sine at $x_1 = 0$ is 1, and at $x_n = \pi/2$ is 0. Enter these derivative values in the fields for DERIV 1: and DERIV n: respectively, and press F6-OK. The rest of the input is the same as it was for the free spline. Repeat the steps to enter that data. If we now approximate the sine at the midpoint of each input interval, we get the values shown in the table below.

x in radians	Approximation of sin(x)	Actual sin(x)
$\pi/24 = 0.131$	0.131	0.131
$\pi/8 = 0.393$	0.383	0.383
$5\pi/24 = 0.654$	0.608	0.608
$7\pi/24 = 0.916$	0.793	0.793
$3\pi/8 = 1.178$	0.924	0.924
$11\pi/24 = 1.440$	0.991	0.991

In this case we see that the clamped spline approximation agrees with the correct value up to three decimal places for all the inputs. We should not count on such accuracy for all functions, but we do get it for the sine function. The important point is that we can generally expect better accuracy from clamped splines than from natural splines if we have the derivatives at the end points, or at least reasonable approximations for those derivatives. If you have 3 or more nodes and they are equally spaces with $h = x_{k+1} - x_k$ then the following give reasonable approximations for the derivatives at the ends. The derivative at the left end is approximately $\frac{1}{h}\left(-\frac{3}{2}y_1 + 2y_2 - \frac{1}{2}y_3\right)$ and at the right end is approximately $\frac{1}{h}\left(\frac{1}{2}y_{n-2} - 2y_{n-1} + \frac{3}{2}y_n\right)$. Using these approximations and the data from Table 1, we have that the derivative of the sine at the left end is approximately 1.024 and at the right end is approximately .004. The approximations we now get for the sine are:

x in radians	Approximation of sin(x)	Actual sin(x)
π/24 = 0.131	0.132	0.131
π/8 = 0.393	0.382	0.383
5π/24 = 0.654	0.608	0.608
7π/24 = 0.916	0.793	0.793
3π/8 = 1.178	0.924	0.924
11π/24 = 1.440	0.991	0.991

Finally, suppose you wish to go back to the free spline for the sine function. Press F1-INPT, then press F6-OK with "Load existing spline" selected. All of the existing splines will be displayed in the menu; it may be necessary to press NXT is you have created more than 6 and the one you want is not showing. Press the function key corresponding to FSIN then press ENTER and you are ready to start approximating the sine function using the free spline you created earlier.

WARNING 1. The number of decimal places in the approximations can never be greater than the number of decimal places in the *y* values that were entered. In this case we entered data with three decimal places with the calculator set to FIX 3. If you were to change the display on the calculator to more than 3 decimal places and approximated a value of the sine, you will see more decimal places displayed, but those numbers past the third decimal place are meaningless.

WARNING 2. Cubic splines should never be used to extrapolate, only to interpolate; that is, only to approximate values between x_1 and x_n. This program is designed to protect the user from making the error of using the spline to extrapolate. It you enter an *x* value less x_1 or greater than x_n you will get the error message "Input out of range".

WARNING 3. The error bound for cubic splines is proportional to $\max_{1\leq j\leq n-1}\left(x_{j+1} - x_j\right)^4$. This suggest one should try to keep the intervals between nodes small. To see what can happen when this rule is violated, repeat the clamped spline example above using the degrees for the input, 0º, 15º, …, 90º. If you now approximate the sine of 7.5º your answer is 2.467, which is not a very impressive approximation for the sine of anything. Clearly, it is important to keep the nodes as close together as possible.

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