

# HP Prime AP Calculus Summer Institute

Materials by Mark Howell













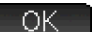

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## Getting Acquainted With HP Prime





HP Prime is a color, touchscreen graphing calculator, with multi-touch capability, a Computer Algebra System (CAS), an Advanced Graphing app that lets you graph any relation in two variables (graphing something like  $\sin(xy) = \cos(xy)$  for example), and a dynamic geometry app. In this section, we'll take a look at how to find your way around HP Prime, and get acquainted with the Function app.

First, here are a few conventions we'll use in this document:





- A key that initiates an unshifted function is represented by an image of that key:  
, ,  and so on.
- A key combination that initiates a shifted function (or inserts a character) is represented by the appropriate shift key ( or ) followed by the key for that function or character:  
 initiates the natural exponential function and  inserts the letter F.
- The name of the shifted function may also be given in parentheses after the key combination:  
 (Clear),  (Plot Setup)
- A key pressed to insert a digit is represented by that digit:  
5, 7, 8, and so on.
- All fixed on-screen text—such as screen and field names—appear in bold:  
**CAS Settings**, **Xstep**, **Decimal Mark**, and so on.
- A menu item selected by touching the screen is represented by an image of that item:  
, , , and so on.




**NOTE:** You must use your finger to select a menu item, or navigate to the selection and press









- Cursor keys are represented by , , , and . You use these keys to move from field to field on a screen, or from one option to another in a list of options.







The ON-OFF key is at the bottom left of the keyboard. When a new HP Prime is turned on for the first time, a "splash" screen appears that invites the user to select a language and to make some initial setup choices. For most users, accepting the default options is the way to go.

The screen brightness can be increased by pressing and holding  and  or decreased by pressing and holding  and .








Take a minute to look at the layout of the keyboard. The top group of keys, with the black background, are primarily for navigating from one environment to another. Pressing  takes you to the home calculation screen, and pressing  takes you to a similar calculation environment for doing symbolic or exact computations. Pressing  takes you to a menu where you can select from all the applications in the HP Prime, like **Function** or **Parametric** or **Geometry**. The bottom group of



keys is mainly for entering or editing mathematical expressions. There are also environments for entering lists,   , matrices,   , and user programs,   .

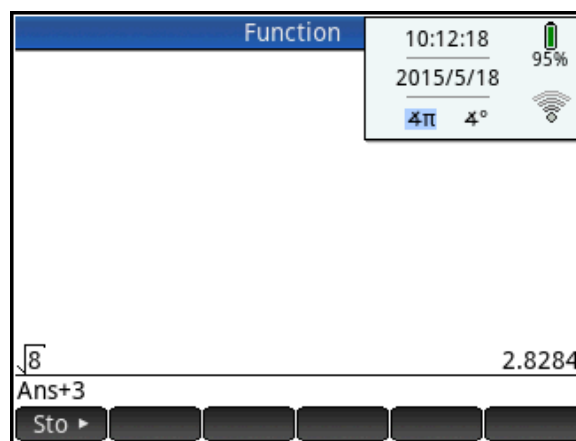
Some care was taken when deciding where to place certain keys. The number  $\pi$  for instance, is   . The list delimiters, {}, appear just to the right of the LIST key,   and the matrix delimiters, [], appear just to the right of the MATRIX key,   .


Things you can do in both CAS and Home views:





- Tap an item to select it or tap twice to copy it to the command line editor
- Tap and drag up or down to scroll through the history of calculations
- Press  to retrieve a previous entry or result from the other view
- Press the Toolbox key () to see the Math and CAS menus as well as the Catalog
- Press  to open a menu of easy-to-use templates
- Press  to exit these menus without making a selection
- Tap  ,  , and  menu buttons once to activate

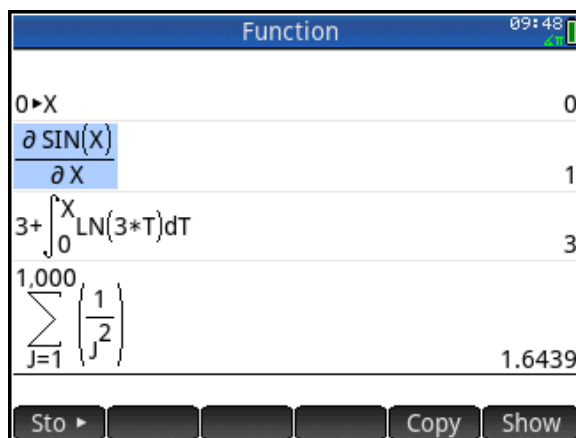
## Home View

Turn on your HP Prime and take a look at the different sections of the screen in the **HOME** view. The top banner across the top is called the Title Bar, and it tells you what operating environment you are currently working in (like **HOME** or **FUNCTION SYMBOLIC VIEW**). If you press a shift key, an annunciator comes on at the left of the Title Bar. On the right, you see a battery level indicator, a clock, and the current angle mode. You can tap this Quick Settings section at the top right to see a calendar (by tapping the date and time), connect to a wireless classroom network (by tapping the wireless icon), or change the angle mode (by tapping the angle mode indicator).



The middle section of the **HOME** view contains a history of past calculations. You can navigate through the history with the cursor keys, or using your finger to select (by tapping) or scroll (by swiping). The edit line is just below the history section. This is where you enter mathematical expressions to evaluate numerically. At the bottom are the menu keys, consisting of  in the **HOME** view. These menu keys are context sensitive. Their labels and function changes depending on what environment you're in.

Home view is for numerical calculations, as the next few examples show. First, store 0 in the real variable  $X$  by entering 0   $X$ . To enter the variable  $X$ , press   (or ).



With 0 stored in X, press  $\left[ \frac{\square}{\square} \right]$  and choose the derivative template. In the numerator, enter  $\left[ \frac{\sin}{\sin} \right]$   $\left[ \frac{\alpha}{\alpha} \right]$   $\left[ \frac{x}{x} \right]$ ; then tap on the denominator and enter  $\left[ \frac{\alpha}{\alpha} \right]$   $\left[ \frac{x}{x} \right]$ . Press  $\left[ \frac{\text{Enter}}{\text{Enter}} \right]$  to see the result: the value of the derivative of  $\sin(X)$  (which is  $\cos(X)$ ), when  $X=0$  is 1. For another example, enter  $3+$ , then return to the Template menu and select the integral template. Let the integrand be  $\text{LN}(3 \cdot T)$  and the limits of integration be from 0 to X, as shown in the figure. Press  $\left[ \frac{\text{Enter}}{\text{Enter}} \right]$  to see the result, which can be seen by inspection to be 3, since  $X=0$ . The figure to the right also shows a summation. In all of these cases, the results evaluate to a real number. In Home view, all results evaluate to a real or a complex number, or a matrix, list, etc. of real and/or complex numbers. You can tap on any previous input or result in the history to select it. When you do, two new menu buttons appear: **Copy** and **Show**. The former copies the selection to the cursor position while the later typesets the selection in textbook format in full-screen mode.

## CAS View

CAS view, on the other hand, is for symbolic or exact numerical results. Press  $\left[ \frac{\text{CAS}}{\text{CAS}} \right]$  to open CAS view. Let's repeat our derivative calculation. Press  $\left[ \frac{\sin}{\sin} \right]$   $\left[ \frac{\alpha}{\alpha} \right]$   $\left[ \frac{x}{x} \right]$ ; then tap on the denominator and enter  $\left[ \frac{\alpha}{\alpha} \right]$   $\left[ \frac{x}{x} \right]$ . Press  $\left[ \frac{\text{Enter}}{\text{Enter}} \right]$  to see the result: the symbolic derivative of  $\sin(x)$  is  $\cos(x)$ . Note the use of lower-case variables here. Similarly,  $3 + \int_0^x \text{LN}(3 \cdot t) dt$  evaluates to  $x \cdot \text{LN}(3 \cdot x) - x + 3$ . The CAS can also use  $\infty$ , as

| CAS Function   |  |
|--|--|
| $\frac{d}{dx} \sin(x)$                                     | $\cos(x)$                              |
| $3 + \int_0^x \text{LN}(3 \cdot t) dt$                     | $x \cdot \text{LN}(3 \cdot x) - x + 3$ |
| $\lim_{n \rightarrow \infty} \cos\left(\frac{1}{n}\right)$ | 1                                      |
| $\sum_{j=1}^{\infty} \left(\frac{1}{j^2}\right)$           | $\frac{\pi^2}{6}$                      |

shown in the evaluation of  $\lim_{n \rightarrow \infty} \cos\left(\frac{1}{n}\right)$  and  $\sum_{j=1}^{\infty} \left(\frac{1}{j^2}\right)$ , as shown in the figure.

## Copy and Paste

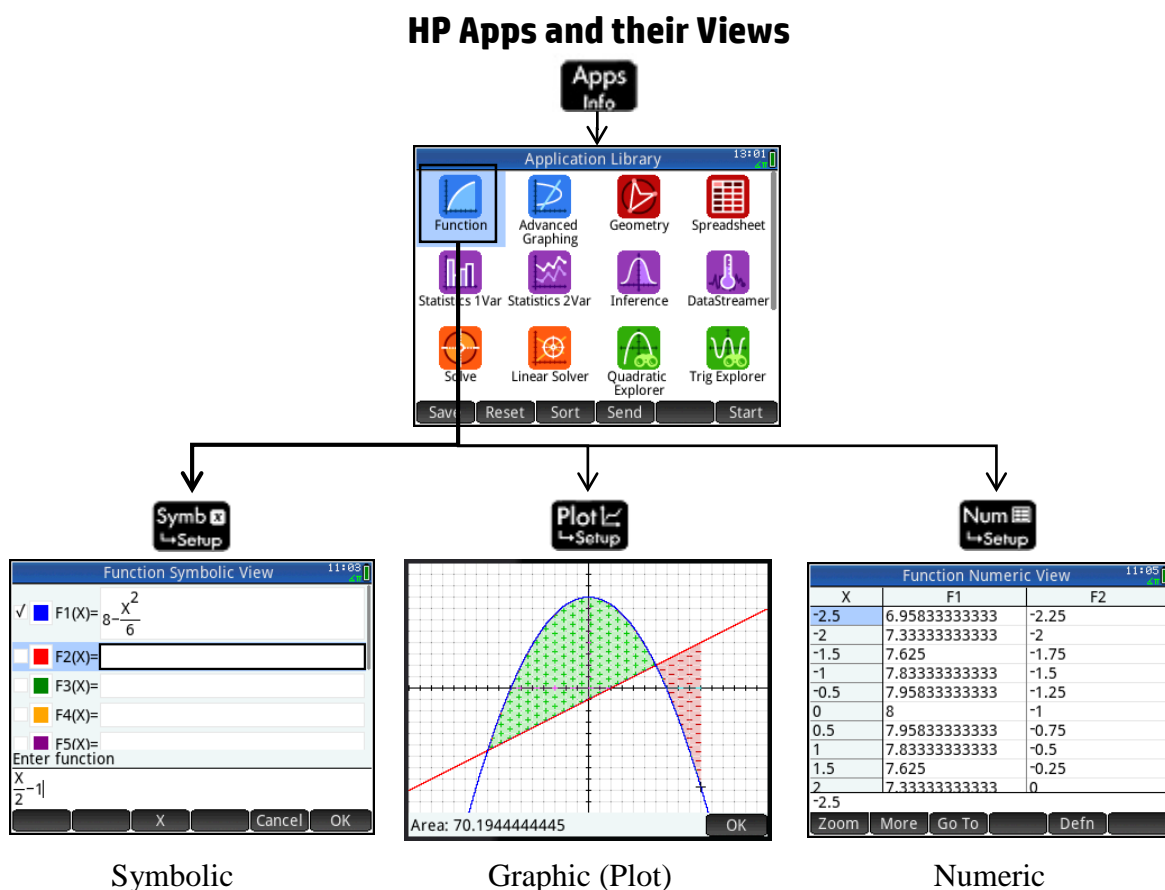
As mentioned earlier, both the CAS and Home view histories use **Copy** to copy the selection to the command line. There is also **Shift**  $\left[ \frac{\text{View}}{\text{Copy}} \right]$  (Copy) and **Shift**  $\left[ \frac{\text{Menu}}{\text{Paste}} \right]$  (Paste) that copies the selection to the Prime clipboard and pastes from that clipboard to the cursor position. This functionality makes it possible to copy and paste from one environment to another anywhere in your HP Prime. With data, you can tap and hold, then drag to select a rectangular array of cells, then copy and paste anywhere else. With the HP Prime Virtual Calculator, you can copy an array of cells in a spreadsheet on your PC and paste the numerical data anywhere in the Prime Virtual Calculator. You can then send the data to another HP Prime.

## Delete and Clear

In both CAS and Home views, you can select an item in history and press  $\left[ \frac{\text{Del}}{\text{Del}} \right]$  to delete it. Press **Shift**  $\left[ \frac{\text{Esc}}{\text{Clear}} \right]$  (Clear) to delete the entire history. If you are in any view (Symbolic, Plot Setup, etc.), **Shift**  $\left[ \frac{\text{Esc}}{\text{Clear}} \right]$  (Clear) will reset (clear) all settings in the current page of the view back to their factory defaults.

## HP Prime Apps and Their Views

The HP Prime graphing calculator comes pre-loaded with a number of apps. Each app was designed to explore an area of mathematics or to solve problems of a specific type. Every Prime app is divided into one or more views. Most commonly, an app has a Symbolic view, a Plot (or Graphic) view, and a Numeric view. In this sense, the apps all have a common structure so they are easier to learn to use as a set of apps. The Prime app schema is shown in the figure below.



Press **Apps Info** to open the App Library. Tap on an app to start it, or navigate the library using the cursor pad and tap **Start** to launch the app.

Fill the app with data while you work, and save it with a name you'll remember. Then reset the app and use it for something else. You can come back to your saved app anytime—even send it to your colleagues! HP Apps have app functions as well as app variables; you can use them while in the app, or from the CAS view, Home view, or in programs.

# Limits, Asymptotes, and Zooming

## HP Prime Functionality Introduced:

Using the Function App Symbolic, Plot, and Numeric views; zooming in to a graph; zooming in to a table

## AP Calculus Content:

Limits at a point and at infinity; one-sided limits; horizontal and vertical asymptotes

## Activity:

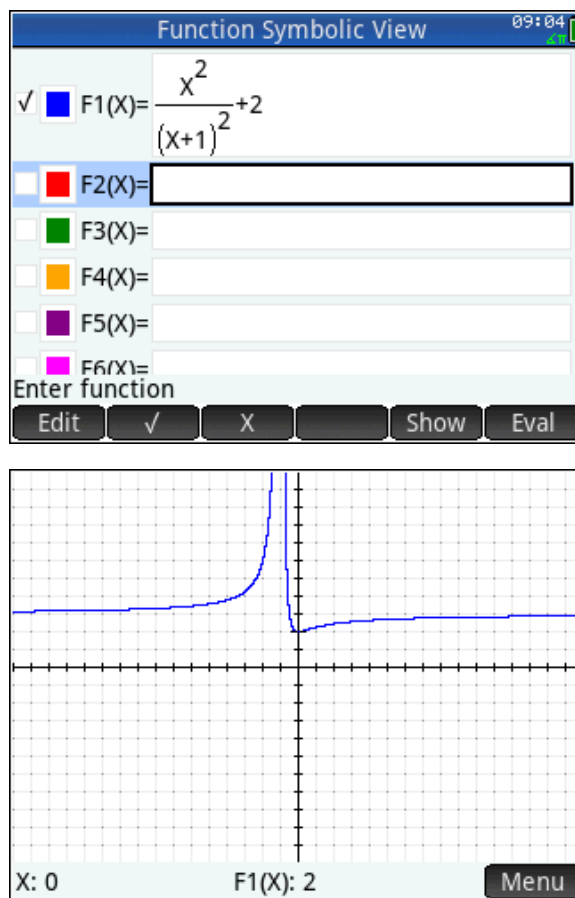
Press **Apps** and select the Function App. Press

**Shift** **Esc** (Clear) and enter  $F1(x) = \frac{x^2}{(x+1)^2} + 2$ , as

shown.

Here, you will investigate the behavior of this function in its graphic and numeric representations, as the values of  $x$  grow without bound.

First, press **View** and select 4 to see the graph in the decimal view. In this view, pixels are 0.1 units horizontally and vertically. You should see the graph at right.



With HP Prime, it is a simple matter to look at the graphic behavior for large values of  $x$ . Simply swipe your finger horizontally from right to left on the graph to scroll to the right. Tap the graph after it appears to become horizontal, and the input and output will be displayed at the input where you tap. Enter the values of **X** and **F1(X)** you see.


1. X \_\_\_\_\_ F1(X) \_\_\_\_\_

Swipe your finger a few more times in the same manner and again record the values of **X** and **F1(X)**

2. X \_\_\_\_\_ F1(X) \_\_\_\_\_

3. You should see that the values of  $F1(X)$  are pretty close to a positive integer. What is that integer? \_\_\_\_\_

Now let's look at this behavior numerically in a table of values.

Press  to see a table of inputs and outputs for F1.

There are several ways to look at the outputs from F1(X) for large values of X. One is to simply enter a value for X. With the cursor anywhere in the column for X, enter 10. The table immediately adjusts to the new value.

Enter 100, then 1000, then 1000000 for X, and observe each time what happens to the outputs from F1(X). You should see them approaching the same integer you gave as your answer to question 3.




In mathematical language, we say that the limit of F1(X) as x approaches positive infinity is 3, and express that result with this notation:

$$\lim_{x \rightarrow \infty} F1(x) = 3.$$

We also say that the line  $y = 3$  is a *horizontal* asymptote for the graph of  $y = \frac{x^2}{(x+1)^2} + 2$ . Notice

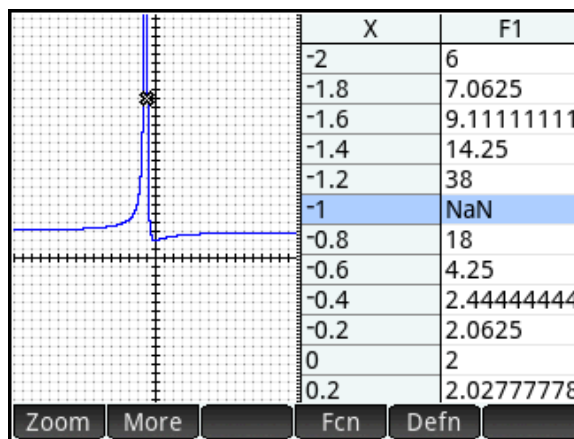
that we can make the values of F1(X) as close as we want to 3 (within the limitations of the numerical precision of the calculator) by making the inputs, X, sufficiently large.

You can also use a horizontal pinch to zoom horizontally. Place two fingers together horizontally in Plot view, pause a moment, and then move them apart horizontally to zoom in on just the x-values. You will see a pair of blue horizontal arrows indicating that horizontal zoom is active.

Let's take a look at using HP Prime to investigate the behavior of the outputs from F1(X) for values of X close to -1. Press  and select 4 to restore the decimal view. Then simply press  1 to move the cursor to a point with an x-coordinate of -1. This time, we'll use the **Split Screen: Plot-Table** View to explore the function's behavior. With the cursor on a point with x-coordinate -1, press  2 to select the **Split Screen: Plot-Table** view.

| Function Numeric View |               | 15:41      |
|-----------------------|---------------|------------|
| X                     | F1            |            |
| 0                     | 2             |            |
| 0.1                   | 2.00826446281 |            |
| 0.2                   | 2.02777777778 |            |
| 0.3                   | 2.05325443787 |            |
| 0.4                   | 2.08163265306 |            |
| 0.5                   | 2.11111111111 |            |
| 0.6                   | 2.140625      |            |
| 0.7                   | 2.16955017301 |            |
| 0.8                   | 2.1975308642  |            |
| 0.9                   | 2.2243767313  |            |
| 0                     |               |            |
| Zoom                  | More          | Go To Defn |

| Function Numeric View |               | 15:42      |
|-----------------------|---------------|------------|
| X                     | F1            |            |
| 10                    | 2.82644628099 |            |
| 10.1                  | 2.82793604415 |            |
| 10.2                  | 2.8294005102  |            |
| 10.3                  | 2.83084031639 |            |
| 10.4                  | 2.83225607879 |            |
| 10.5                  | 2.8336483932  |            |
| 10.6                  | 2.83501783591 |            |
| 10.7                  | 2.83636496457 |            |
| 10.8                  | 2.83769031887 |            |
| 10.9                  | 2.8389944213  |            |
| 10                    |               |            |
| Zoom                  | More          | Go To Defn |





4. The output for  $x = -1$  appears as **NaN** (calculator-speak for "not-a-number"). Why is there no output at  $x = -1$ ? \_\_\_\_\_

Press **Zoom**, and select **5 XIn**. This changes the horizontal scale only, leaving the vertical scale unchanged. Notice that the values in the table adjust as you zoom in. Press **Zoom**, and select **5 XIn** several more times. If necessary, swipe vertically on the graph to keep it in view. You can also use the horizontal pinch gesture described above to zoom in. What appears to be happening with the function outputs as the inputs get close to  $-1$ ? What is happening with the graph?

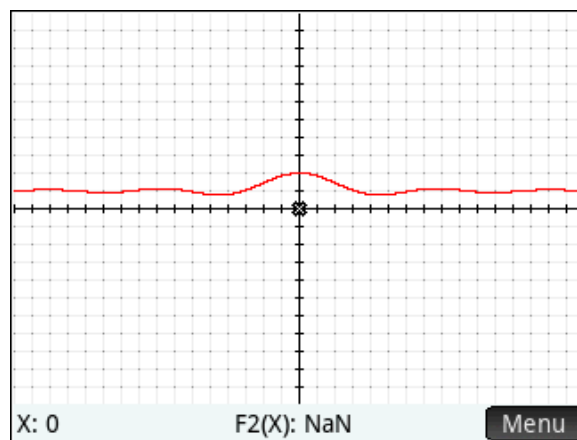
You can also zoom in vertically using a pinch gesture. Place two fingers vertically in Plot view, pause a moment, and move them apart to zoom in. You will see two vertical blue arrows indicating that a vertical zoom is active.

Here, we say that  $\lim_{x \rightarrow -1} F1(x) = +\infty$ . (We might also say that the function has no limit as  $x$  approaches  $-1$ .) The values of the function increase without bound as the inputs approach  $-1$ . We also say that the line  $x = -1$  is a *vertical* asymptote for the graph of  $y = \frac{x^2}{(x+1)^2} + 2$ .

From this example, the idea of limit might seem simple enough. But it can be quite slippery, as the next example illustrates. Press **Symb**, and tap the check mark to the left of  $F1(X)$  to deselect it. Then

enter  $F2(x) = \frac{\sin(x)}{x} + 1$ . Again, press **View** and

select 4 to see the graph in the decimal view. You should see the graph at right. Proceed as you did for the previous example, and scroll to the right to see what happens to the values of  $F2(X)$  as  $x$  gets large.



5. What is  $\lim_{x \rightarrow +\infty} \left( \frac{\sin(x)}{x} + 1 \right)$ ? \_\_\_\_\_

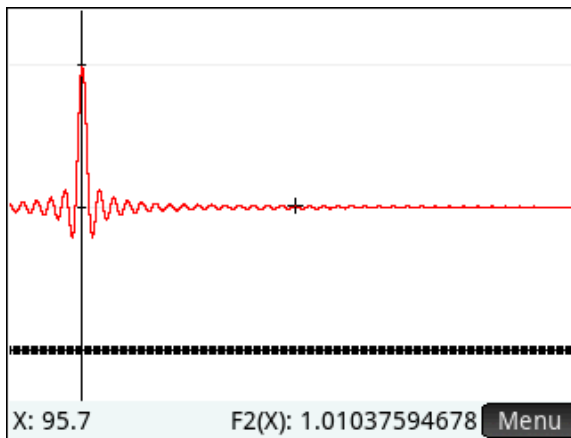
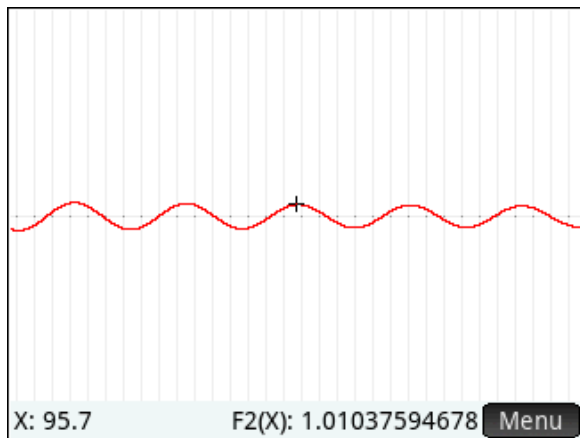
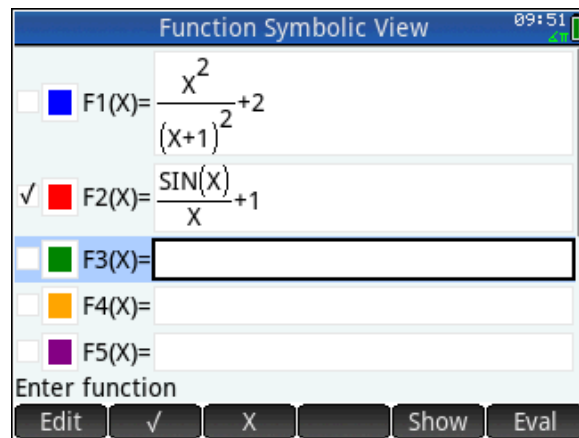
After you have scrolled to the right and the graph appears to be horizontal, tap on the graph, press the **Menu** key and then press **Zoom**, and select **7 Yin**. You can also use a vertical pinch gesture. Place two fingers vertically in Plot view and move them apart vertically.

This leaves the horizontal scale unchanged, but zooms in vertically. Zoom in vertically several times, until you can see that the graph is not really horizontal at all! Zoom out (**Zoom 4 Out**) a few times to get a better idea of what's going on. You should see that the amplitude of the oscillations is getting smaller as the value of  $x$  get larger. This example illustrates why, even though

$$\lim_{x \rightarrow +\infty} \left( \frac{\sin(x)}{x} + 1 \right) = 1, \text{ we can't say that the values of}$$

the function are "getting closer and closer to 1". It is more appropriate to say something like, "We can

make the values of  $\frac{\sin(x)}{x} + 1$  as close as we want to 1 by making the values of  $x$  sufficiently large."



Press **Num** and enter 0 into any cell in the X column. We can zoom in to the table in the same way we zoomed into a graph. With 0 selected in the X column, press **Zoom** and select **1 In**. Notice that the step size changes in the table. Press **Ans**, a shortcut for zooming in, several times. Of course, you can also use a vertical pinch gesture to zoom in or out on a row in the table.

6. What is  $\lim_{x \rightarrow 0} \left( \frac{\sin(x)}{x} + 1 \right)$ ? \_\_\_\_\_

## Extension:

An HP Prime App called "Limits" contains definitions for these, and other, functions. Collectively, these examples illustrate just about every type of limit problem you might encounter. Distribute the Limits App to students. Then have them investigate each function at the indicated point, and make a conclusion for each using the language of limits.

$$F3(x) = \frac{x^2 - 4}{x - 2}; x = 2$$

$$F4(x) = \sin\left(\frac{1}{x}\right); x = 0$$


$$F5(x) = x \cdot \sin\left(\frac{1}{x}\right); x = 0$$

$$F6(x) = \arctan\left(\frac{1}{x}\right); x = 0; \text{ Use the language of one-sided limits}$$

$$F7(x) = \frac{1}{\pi} \arctan\left(\frac{1}{x^2}\right); x = 0$$

$$F8(x) = \frac{|x - 2|}{x - 2}; x = 2; \text{ Use the language of one-sided limits}$$

$$F9(x) = \begin{cases} \frac{x^2}{4}, & \text{if } x \leq 2 \\ 4 - x, & \text{if } x > 2 \end{cases}; x = 2; \text{ Use the language of one-sided limits}$$

Note: you can add a third branch to a piecewise-defined function. Place the cursor at the end of the second branch and press .

$$F0(x) = \frac{2^{-x+2} + 3^x}{2^{-x} - 5 \cdot 3^x}; x \rightarrow +\infty \text{ and } x \rightarrow -\infty; \text{ Also describe the horizontal and vertical asymptotes}$$

## Answers

1. Answers will vary.  $X = 61$ ,  $F1(X) = 2.968$
2. Answers will vary.  $X = 146$ ,  $F1(X) = 2.98644$
3. 3
4. Trying to evaluate  $F1(1)$  results in division by 0
5. The graph is becoming close to vertical.
6. 1
7. 2



# Introducing the Derivative

## HP Prime Functionality Introduced:

Drawing a tangent line to the graph of a function; using **STO>** to store the X and Y coordinates of the trace cursor on the graph into variables; evaluating a difference quotient on the **HOME** screen

## AP Calculus Content:

Local linearity; differentiability; limit of an average rate of change to get the slope of a curve at a point; introduction of the derivative at a point

### Activity:

Press **Apps Info** and select the Function App. Press **Shift Esc** (Clear), and enter  $F1(x) = \sin(x)$ . Press **View Copy** and select 4 to see the graph in the decimal view. Press **Ans +**, a shortcut for zooming in, several times, until the graph keeps the same shape.

1. What happened to the shape of the graph as you zoomed in?

---

Make sure the cursor is on the point where  $X = 0$ . Press **Menu**, and then press **Fcn**. Select **5 Tangent** to draw the tangent line to the graph of  $F1$  at  $X = 0$ .

2. What appears to be true about the graph of  $F1(x) = \sin(x)$  and its tangent line at  $X = 0$ ?

---

Press **Base -** a few times until you can see a difference between the tangent line and the graph of  $F1(x) = \sin(x)$ . It appears that an equation for the tangent line is  $y = x$ . We could use the  $y$ -coordinate of a point on the tangent line to approximate the  $y$ -coordinate of a point on the graph of  $F1(x) = \sin(x)$ .

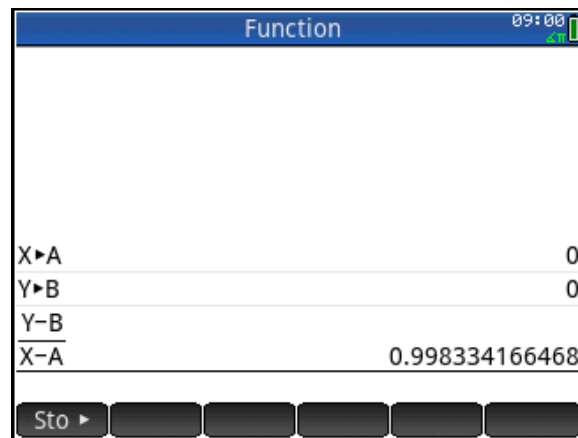
3. When  $x = 0.2$ , what is the  $y$ -coordinate on the tangent line? \_\_\_\_\_

Press **Menu**, then enter 0.2 to move the cursor to the point on the graph of  $F1(x) = \sin(x)$  where  $x = 0.2$ .

4. What is  $\sin(0.2)$ ? \_\_\_\_\_ How big is the error if we used the tangent line to approximate  $\sin(0.2)$ ? \_\_\_\_\_ (The answer comes from subtracting your answer to the first part of question 4 from your answer to question 3.)

Press and select 4 to see the graph in the decimal view. Press . Now you will store the cursor coordinates, which HP Prime keeps internally in the variables X and Y, into the variables A and B.

Enter X A , and Y B to do this. Press , then press to move the cursor one pixel to the right of X = 0. Press , and enter the expression  $\frac{Y - B}{X - A}$ .



This expression,  $\frac{Y - B}{X - A}$ , is called a difference quotient (it is the quotient of two differences), and represents the slope of the line segment joining the points  $(X, \sin(X))$  and  $(A, \sin(A))$ . It also represents the average rate of change of  $f(x) = \sin(x)$  on the closed interval  $[A, X]$ . In this case,  $A = 0$ .

Now you will zoom in to the graph and recalculate this average rate of change. Press , then press to move the cursor back to the point where X = 0. Press one time to zoom in, then press to move the cursor one pixel to the right of X = 0. Press . Now, all you have to do is press to reevaluate the difference quotient with the new point. Repeat this process several times: go back to the graph, move the cursor to X=0, zoom in, move the cursor one pixel to the right of X=0, go **HOME**, and recalculate the difference quotient. You should observe two important and related things going on: the graph straightening out, and the difference quotients converging to a limit.

Now, press and select 4 to get back to the decimal view. Press several times to move the cursor to the *left* of X=0. Before you calculate the difference quotient, stop and think!





5. Should the slope be less than or greater than 1? \_\_\_\_\_

Press , then press to calculate the new difference quotient. Were you right?

Now repeat the process of calculating several difference quotients while zooming in to the graph, except this time, use X values that are *less* than 0. You should again see the graph straightening out, and the difference quotients approaching a limit.








6. What is the limit of the average rates of change as X gets close to 0? \_\_\_\_\_




This limit is called the derivative of  $\sin(x)$  at the point where  $x = 0$ , and represents the slope of the tangent line to the graph at that point. When this limit exists, we say that the function is *differentiable* at the point.

Not every function has a graph that looks locally linear. Press  , and tap the check mark to the left of F1(X) to deselect it. Then enter  $F2(x) = 3 - |x + 2|$ . Press  to find the template for absolute values. Press  and select 4 to see the graph in the decimal view.

Use the cursor keys to move to the point where  $X = -2$ . Then zoom in to the graph several times.

7. Does the graph straighten out? \_\_\_\_\_

With the cursor on the point  $(-2, 3)$ , press . Enter  $X$    A and  $Y$    B to store these values into A and B. Press , then press  to move the cursor one pixel to the right of  $X = 0 - 2$ .

Press , and tap the expression  $\frac{Y - B}{X - A}$  to select it, then press  followed by .

8. What is the slope of the line segment when

$X > -2$ ? \_\_\_\_\_

Go back to the graph, move the cursor to the left of  $X = -2$ , return **HOME** and recalculate the difference quotient.

9. What is the slope of the line segment when  $X < -2$ ? \_\_\_\_\_

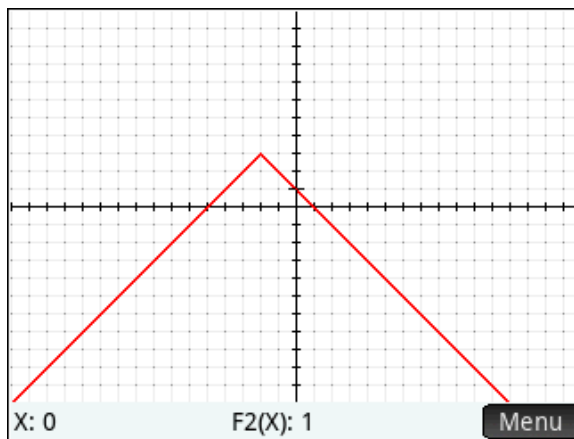
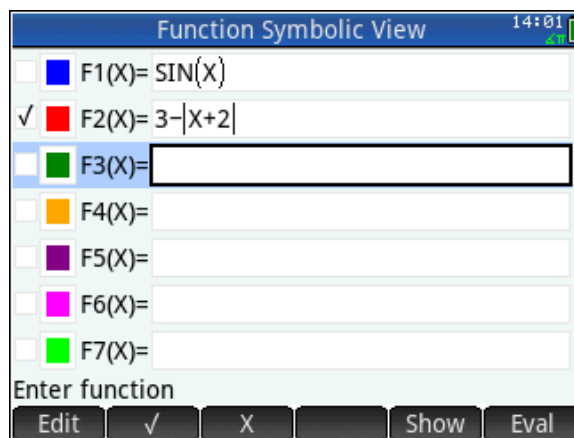
In this case, since the slopes on either side of  $X = -2$  are not the same, we say that the function is *not differentiable* at  $X = -2$ .

Now, enter this function into the Function Symbolic view:  $F3(x) = \sqrt[3]{x^2}$ . Zoom in to the graph at  $X = 0$ .

10. Does the graph appear to be locally linear? \_\_\_\_\_

Using the same procedure outlined in the activity, investigate the slope of the graph of F3(X) at the point where  $X = 0$ .

11. Summarize your findings. \_\_\_\_\_



## Answers

1. As you zoom in, the graph of  $y = \sin(x)$  straightens out.
2. Close to  $x = 0$ , the graphs of  $y = \sin(x)$  and  $y = x$  are almost indistinguishable
3. 0.2
4.  $\sin(0.2) = 0.19866933$ ; the error is  $0.2 - 0.19866933 = 0.00133$
5. The slope should be less than 1
6. 1
7. No
8. -1
9. 1
10. Yes, although the graph is becoming vertical
11. The slope is increasing without bound.



## Sunrise-Sunset Data Activity (Precalculus)





### HP Prime Functionality Introduced:

Distributing an App with the classroom network; using an App to distribute a data set; using the Statistics 2Var App

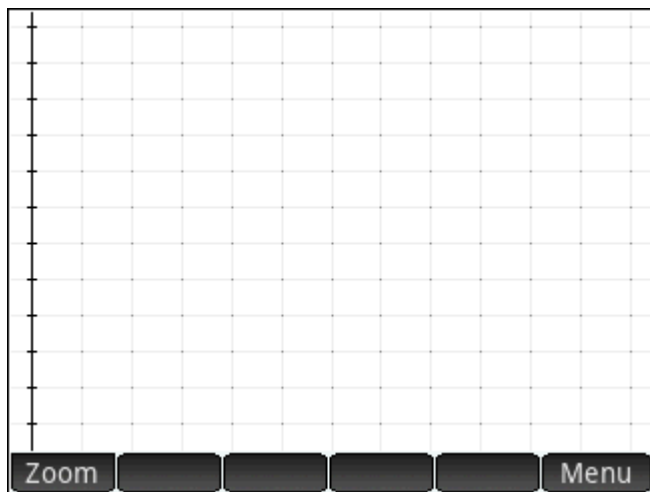
### AP Calculus Content:

There's no Calculus content covered by this activity. It involves fitting a sine curve,  $y = A \sin(B \cdot (x - H)) + K$  to a data set, and interpreting the parameters in context. See the follow-up activity, *Sunrise-Sunset Data Activity (Calculus)*, for a Calculus spin.

### Activity:

Students connect to the HP Classroom Network, and the teacher distributes the SRSS App. Press , and select the SRSS App. The App is based on the built-in Statistics 2Var App. Press  (Info) to see a note that describes what each column of data represents. Press  and  to look at the scatter plot and numeric data, respectively.

Make a rough copy of your scatter plot here:



Answer the following questions using the graph and/or table for reference:

1. On what day was there the most daylight? \_\_\_\_\_
2. On what day was there the least daylight? \_\_\_\_\_
3. During what period of time was the amount of daylight increasing? What seasons are these?  
\_\_\_\_\_  
\_\_\_\_\_

4. During what period of time was the amount of daylight decreasing? What seasons are these?

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5. At what time of year is the amount of daylight increasing the fastest? \_\_\_\_\_

6. At what time of year is the amount of daylight decreasing the fastest? \_\_\_\_\_

Now, you will find an equation in the form  $L(X) = A \sin(B(X-H)) + K$  for the number of minutes of daylight on day  $X$  after the solstice. You'll need to determine the period (which will help you compute the parameter  $B$ ), the amplitude (which will help you compute  $A$ ), the phase shift (which gives  $H$ ), and the vertical shift (which gives  $K$ ). After you find each parameter, on the **HOME** screen, store its value into the variable of the same name.

7. The function  $L(X)$  is periodic. What is the period? \_\_\_\_\_

8. The parameter  $B$  can be found using the equation  $2\pi / B = \text{period}$ . What is the value of  $B$ ?

\_\_\_\_\_ Store this value into the variable  $B$ .

9. The parameter  $A$  is called the amplitude of our function  $L$  and can be found from the equation  $A = (\text{minutes in longest day} - \text{minutes in shortest day}) / 2$ . What is the value of  $A$ ?

\_\_\_\_\_ Store this value into the variable  $A$ .

10. The parameter  $K$  controls the vertical shift of  $L$  and can be found from the equation  $K = (\text{longest day} + \text{shortest day}) / 2$ . What is the value of  $K$ ?

\_\_\_\_\_ Store this value into the variable  $K$ . Can you think of another way to find  $K$ ?

11. The parameter  $H$  controls the horizontal shift of  $L$  and can be found from  $H = 365/4$ . What is

the value of  $H$ ? \_\_\_\_\_ Store this value into the variable  $H$ .

12. Now that you've found  $A$ ,  $B$ ,  $H$ , and  $K$ , you can overlay a graph of  $L$  on your scatter plot. On the Plot screen, press **Menu**, then **Fit** to activate the fit function. Sketch the fit on top of your scatter plot at the top of this activity.
13. Trace on the graph of  $L(X)$  to find the number of minutes of daylight on your birthday (if you are tracing on the Scatter plot, press  $\blacktriangle$  or  $\blacktriangledown$  to trace on the fit). Compare this prediction to online data.  $L(X)$  predicts \_\_\_\_\_; online source says \_\_\_\_\_.

### Extra for the Interested

14. How would the graph of  $L$  change as you move away from the equator towards the North Pole?

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15. What would the graph of  $L$  look like for a place in the Southern Hemisphere at your latitude?

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16. Use a web browser to find sunrise-sunset data for your favorite city. Compare graphs with several other students. If you want, you can set up a scatter plot in S2 of C3 versus C1. The data in C3 are for Honolulu, HI.

17. Does the period of any of the graphs vary? \_\_\_\_\_

18. Does the amplitude vary? \_\_\_\_\_

19. Where are cities that have greater amplitude? \_\_\_\_\_

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20. Sunlight data for Melbourne, AU, a southern hemisphere city with approximately the same latitude as Washington, DC, are in C4. Set up a scatter plot to graph C4 versus C1. How does the graph compare to the graph of C2 versus C1?






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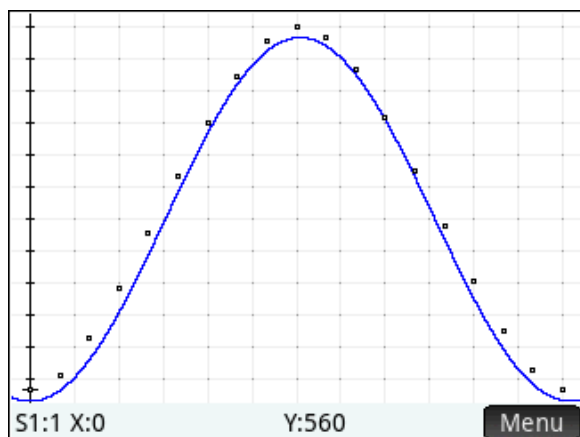
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21. Press . Navigate to the **Fit1** field and press  (Copy). Then navigate to the **Type3** field, and press . Select *User Defined* for the fit type for S3. Then navigate to the **Fit3** field and press  (Paste). Select the expression you just copied. Then edit the expression so that the Fit will work for the Melbourne, AU scatter plot. Hint: You can do this by inserting a single character into the expression! Finally, make sure just S3 is selected for graphing, and that its Fit is selected. Press  and behold!



## Answers

1. Day 180 (June 19) with 901 minutes.
2. Day 0 and day 560 had 560 minutes.
3. From day 0 to day 180 the amount of daylight increases. This is during winter and spring.
4. From day 180 to day 360 the amount of daylight decreases. This is during summer and fall.
5. From the graph, the amount of daylight increases the fastest around day 80. It actually is the steepest at the vernal equinox (around March 21).
6. From the graph, the amount of daylight decreases the fastest around day 280. It actually is the steepest at the autumnal equinox (around September 21).
7. 365 days (or, for the purists, about 365.25 days)
8.  $B = 0.0172142$
9.  $A = 170.5$
10.  $K = 730.5$
11.  $H = 91.25$
- 12.

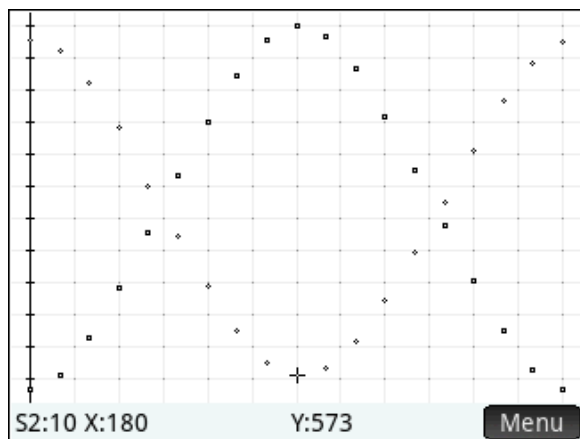


13. Answers will vary, but the predicted time should be close to the actual time.
14. The amplitude increases.
15. It would be a reflection in the line  $y = K$ .
17. The periods of all should be the same (365 days).

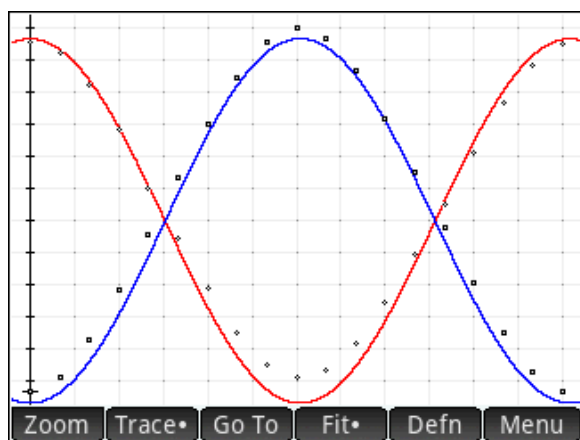
18. The amplitudes may vary.

19. Cities with higher latitudes (closer to the Poles) have higher amplitudes.

20. The graph is a reflection in the line  $y = K$ :



21. The Fit is  $Y = -A \cdot \sin(B \cdot (X - H)) + K$ . Here's the graph:



## Sunrise-Sunset Data Activity (Calculus)

### HP Prime Functionality Introduced:

Using the Statistics 2Var App; using Copy and Paste; using the  $\Delta$ List command

### AP Calculus Content:

This activity involves interpreting rates of change in context; it foreshadows the first derivative test; calculation of many average rates of change; can be revisited after the chain rule is covered to make an interesting connection between the amplitude of the function  $L(X)$  in the previous activity, and the amplitude of the function  $R(X)$  in this activity

### Activity:

Using the data collected from the sunrise-sunset data activity, you will explore some basic calculus ideas. In particular, you will approximate rates of change in the number of minutes of daylight with respect to the number of days since the winter solstice, and look for connections between these values and the particular time of year.

Let the function  $L(X)$  represent the number of minutes of daylight on day  $X$  since the winter solstice,

1. Using the data you collected (not the function you may have found to model the data), approximate the rate of change of  $L(X)$  with respect to  $X$  when  $X = 180$  (approximately halfway through the year). Show your calculations and include units of measure.

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2. What time of the year is it on day 180? What seems to be true about the number of minutes of daylight on a day at this time?

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3. Again using the data you collected, approximate the rate of change of  $L(X)$  with respect to  $X$  when  $X = 270$  (approximately three-fourths of the way through the year). Show your calculations and include units of measure.

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4. What time of the year is it on day 270? What seems to be true about the number of minutes of daylight on a day at this time?


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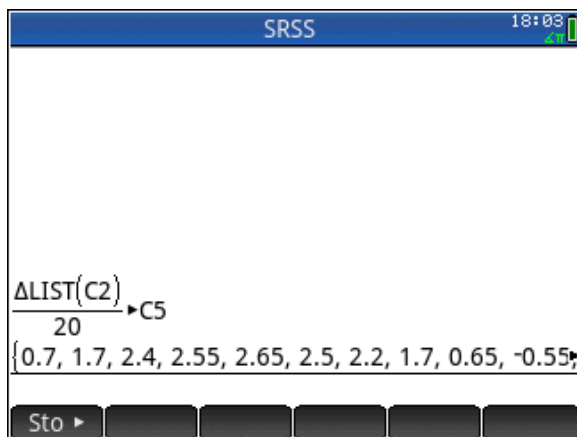
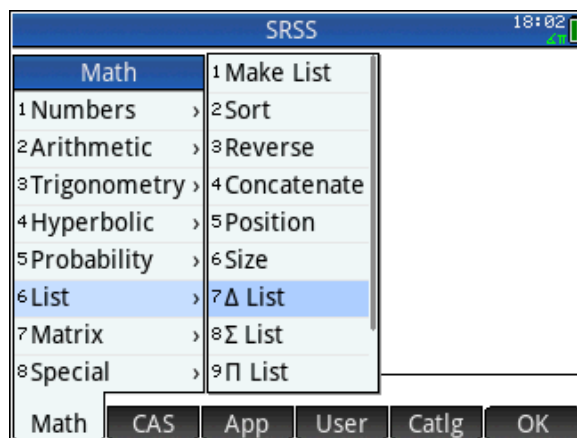


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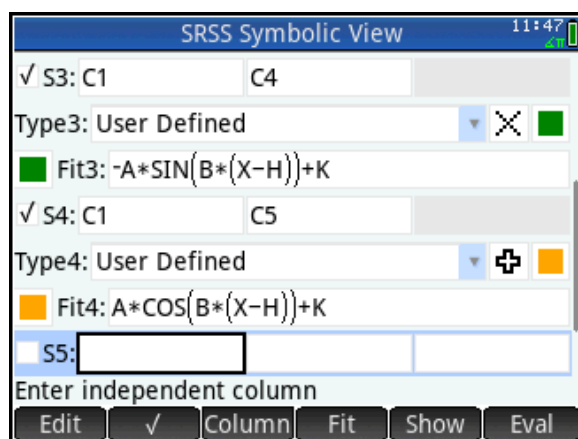


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On the **Home** screen, press  **Math** 6 1 to select the  $\Delta$ List command from the toolbox. Enter the command  $\Delta\text{List}(C2)/20 \text{ STO} \rightarrow C5$ . This stores into  $C5$  the approximate rate of change of  $L(X)$  on all of the days you have daylight data for.



5. Set up and make a scatter plot of C5 versus C1. You'll have to delete the first entry from C1 since there is one fewer difference in C5 than entries in C1.



6. On approximately what day  $X$  does the graph of C5 versus C1 change sign from positive to negative? What time of the year is this?

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7. On approximately what day  $X$  does the graph of C5 versus C1 reach a maximum? What time of the year is this?

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8. On approximately what day  $X$  does the graph of C5 versus C1 reach a minimum? What time of the year is this?

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9. Using the techniques described in the previous sunrise-sunset data activity, determine the equation of a function in the form  $R(t) = A \cos(B(X-H)) + K$  that will pass through the points in the scatter plot of C5 versus C1.

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10. If you did the extensions of the previous activity, describe the similarities and differences among the rates of change of minutes of daylight for DC, Honolulu, and Melbourne, Australia.

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11. If you have covered the chain rule in class, compare the amplitude of this Fit with the amplitude of the Fit of the number of minutes of daylight versus the number of days since the solstice. How are the amplitudes related symbolically?

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## Answers

1. The approximate the rate of change of  $L(t)$  with respect to  $t$  when  $t = 180$  is

$$\frac{890 - 888}{40} = \frac{1}{20} = .05 \text{ or } \frac{890 - 901}{20} = -\frac{11}{20} = -.55 \text{ or } \frac{901 - 888}{20} = \frac{13}{20} = .65 \text{ minutes of daylight per day}$$

2. This is near the summer solstice, the day when the number of minutes of daylight is a maximum..

3. The approximate the rate of change of  $L(t)$  with respect to  $t$  when  $t = 270$  is

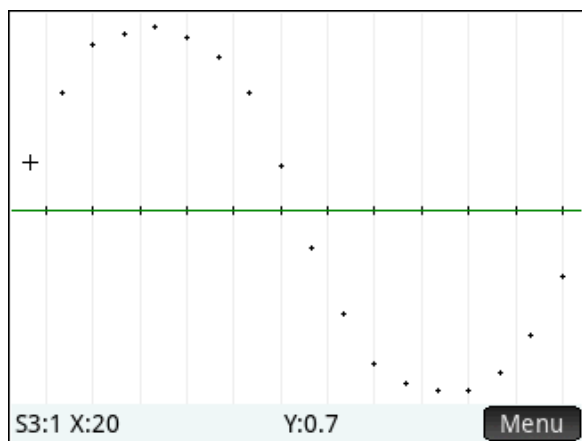
$$\frac{714 - 766}{20} = -2.6 \text{ minutes of daylight per day}$$

4. This is near the autumnal equinox, when there are 12 hours or 720 minutes of daylight. At this time, we lose daylight faster than at any other time of year.

5. The values of C5 are

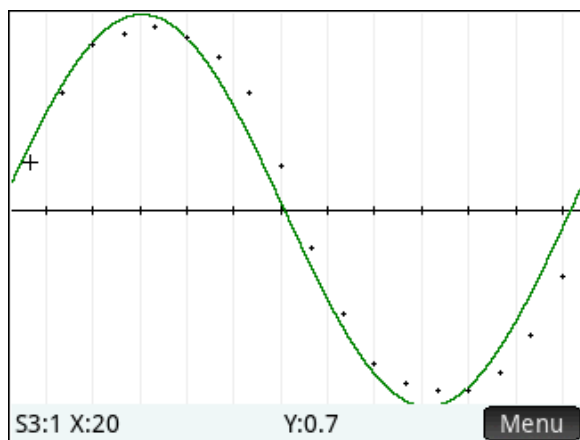
{.7, 1.7, 2.4, 2.55, 2.65, 2.5, 2.2, 1.7, .65, -.55, -1.5, -2.2, -2.5, -2.6, -2.6, 2.35, -1.8, -.95}

Here is the scatter plot:



6. C5 versus C1 changes sign from positive to negative at about day 190. This is close to the summer solstice at approximately day 182.
7. C5 versus C1 reaches a maximum around day 79, near the vernal (spring) equinox at day 90.

8. C5 versus C1 reaches a minimum around day 265, near the autumnal (fall) equinox at day 270.
9.  $K = 0$ . The period is 365 days, so  $B = 2\pi/365$ . The phase shift  $H$  is one quarter of a year, so  $H = 90$ . The amplitude is half the difference between the maximum and minimum values of L5, so  $A = 2.625$ . The resulting model is  $y = 2.625 \sin(2\pi/365(x-90))$ . Overlaying the model on the scatter plot produces this result:



10. The amplitude of the  $R(x)$  function is  $B$  times the amplitude of the  $L(x)$  function.



# The Derivative Function



HP Prime Functionality Introduced:

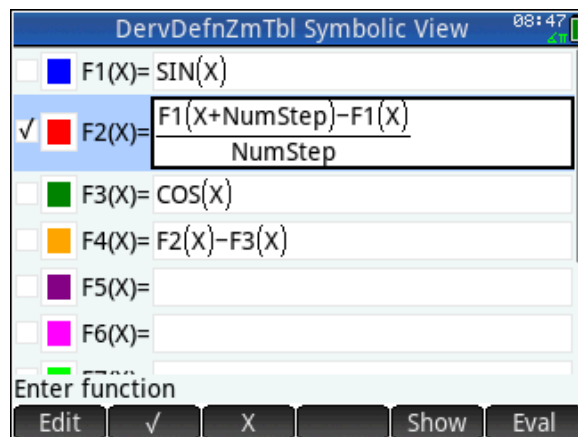
Zooming in to a table; using one function to define another; using Numstep to define a function



AP Calculus Content:


Definition of derivative,  $f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$

## Activity:

Students connect to the HP Classroom Network, and the teacher distributes the DervDefnZmTbl App. Once you have the App, press , and select DervDefnZmTbl. The App is based on the built-in Function App. Press  to see the definitions of the functions you will explore in this activity. Notice that, for the time being, F2(X) is the only function selected.



Press   (Num Setup). There, you see values for variables that control how the Numeric View appears, including NumStart, NumStep, and NumZoom. NumStart is the top value in the table. NumStep is the difference between adjacent inputs in the table, and NumZoom is a factor that NumStep changes by when you zoom in (divide NumStep by NumZoom) or zoom out (multiply NumStep by NumZoom).

1. What is the value of NumStep? \_\_\_\_\_
2. Press  to see the definitions of the functions again. Can you predict what the graph of F2(X) will look like?

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The graph of F2(X) resembles a common trigonometric function. Press  to see it.

3. What trigonometric function does the graph look like?

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4. Explain why the graph of  $F2(X)$  looks the way it does.

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If you had trouble with question 4, consider this question. When  $X = 0$ ,

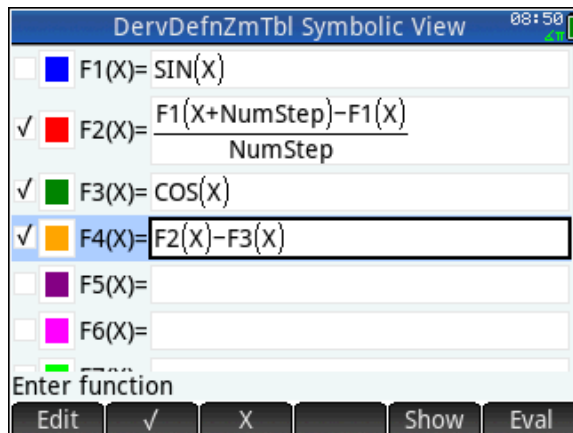
$$F2(0) = \frac{\sin(0 + 0.1) - \sin(0)}{0.1}$$


This equation should look familiar. It represents an approximation for the derivative of a certain function at a certain value of  $X$ .

5. What function? \_\_\_\_\_ What value of  $X$ ? \_\_\_\_\_

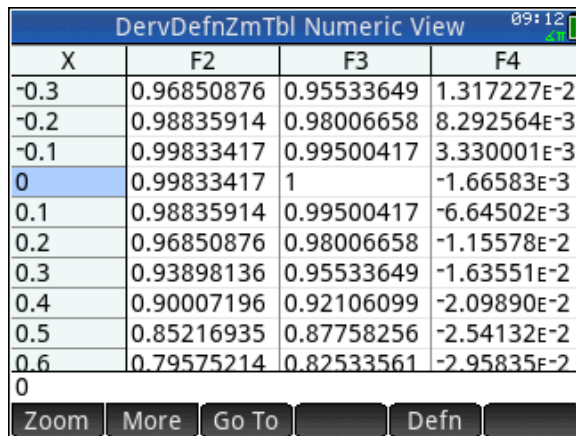
For each value of  $X$ ,  $F2(X)$  represents an approximation for the derivative.

$F2(X)$  is not exactly the derivative, though. Press  and look again at the function definitions. Notice that  $F3(X) = \cos(X)$ , and  $F4(X)$  is the difference between our approximation for the derivative and the exact derivative. Now, you will look at the numeric view for  $F2(X)$ ,  $F3(X)$ , and  $F4(X)$ . First, select  $F3(X)$  and  $F4(X)$  by tapping the check box next to each function in the Symbolic view.



Then press  and study the table. In particular, compare the values of  $F2(X)$  and  $F3(X)$  for the same  $X$ . The values of  $F4(X)$  represent the difference between the approximation of the derivative of  $\sin(X)$  and its exact derivative, for each value of  $X$ .

Now, you will zoom in to the table. Each time you zoom in to the table, the variable Numstep is divided by 4 (the value of Numzoom). This causes the values of  $X$  to get closer together. But remember that the variable Numstep appears in the definition of  $F2(X)$  where you may be accustomed to seeing a variable called  $h$  or perhaps  $\Delta x$ . Therefore, zooming in to the table should make the approximations better,



| X    | F2         | F3         | F4          |
|------|------------|------------|-------------|
| -0.3 | 0.96850876 | 0.95533649 | 1.317227E-2 |
| -0.2 | 0.98835914 | 0.98006658 | 8.292564E-3 |
| -0.1 | 0.99833417 | 0.99500417 | 3.330001E-3 |
| 0    | 0.99833417 | 1          | -1.66583E-3 |
| 0.1  | 0.98835914 | 0.99500417 | -6.64502E-3 |
| 0.2  | 0.96850876 | 0.98006658 | -1.15578E-2 |
| 0.3  | 0.93898136 | 0.95533649 | -1.63551E-2 |
| 0.4  | 0.90007196 | 0.92106099 | -2.09890E-2 |
| 0.5  | 0.85216935 | 0.87758256 | -2.54132E-2 |
| 0.6  | 0.79575214 | 0.82533561 | -2.95835E-2 |

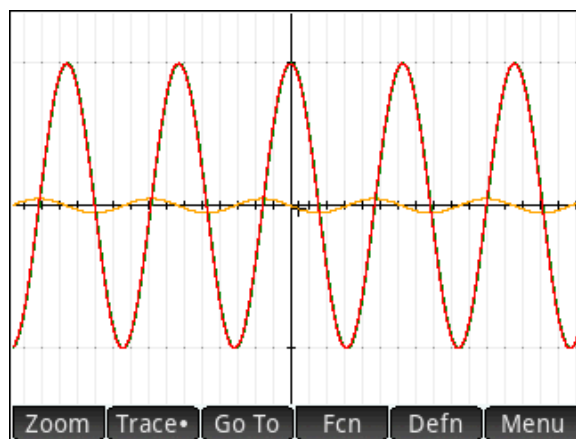
since that process makes Numstep smaller. Press **Zoom** and select **1 In x4**. For comparison, you can look at the table of values on the previous page.

6. At each value of  $X$ , compare the values of  $F2(X)$  and  $F3(X)$ . Are they closer together? \_\_\_\_\_

7. Look at the values of  $F4(X)$ . Are they larger or smaller than before?  
\_\_\_\_\_

Zoom in to the table a few more times, but don't get carried away! You can also press **Ans** as a shortcut to zoom in, just as you can with the graph.

Press **Plot Setup** and zoom in vertically by pinching. Place two fingers together vertically, pause for a moment, and then move them apart vertically. You will see two vertical blue arrows indicating that a vertical zoom is active. Repeat this gesture until you can see that  $F4(X)$  is not really constantly 0. The smaller you make Numstep, the more you have to zoom in vertically to see that  $F2(X)$  and  $F3(X)$  are different.



## Answers

1. NumStep is 0.1
2.  $\cos(X)$
3.  $\cos(X)$
4.  $F2(X)$  is a difference quotient that approximates the derivative of  $\sin(X)$  at each value of  $X$ . Since NumStep is a small number, 0.1, the graph of  $F2$  resembles that derivative.
5.  $\sin(X)$  at  $X = 0$
6. In general, the smaller NumStep is, the closer the values of  $F2(X)$  are to  $\cos(X)$ . That is, the approximation gets better as we zoom in to the table.
7.  $F4(X)$  is the error from using the difference quotient,  $F2(X)$ , to approximate the derivative,  $F3(X)$ . So as we zoom in, the errors get smaller.



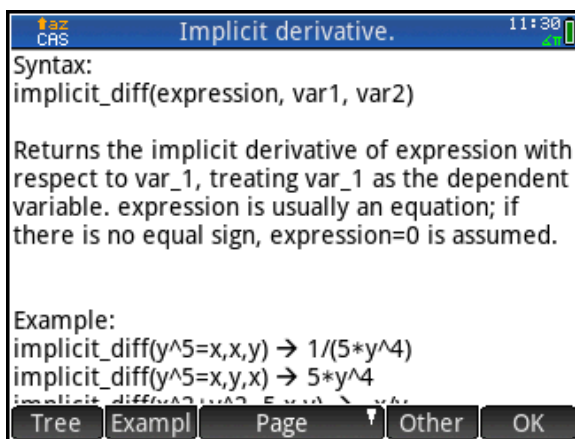
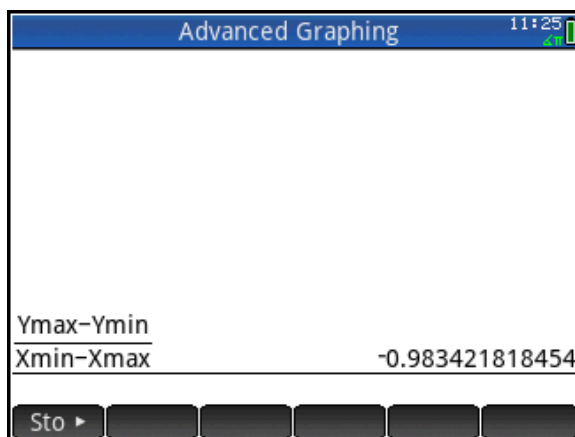
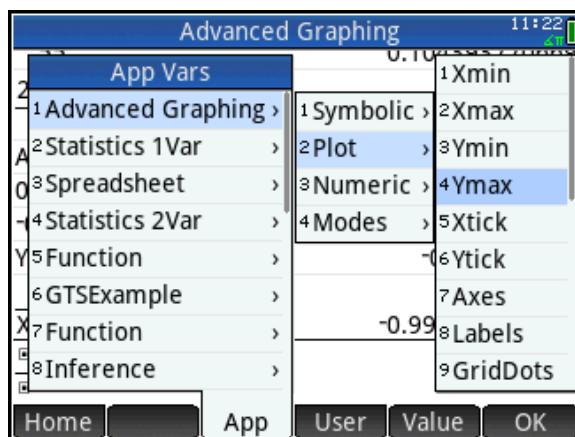
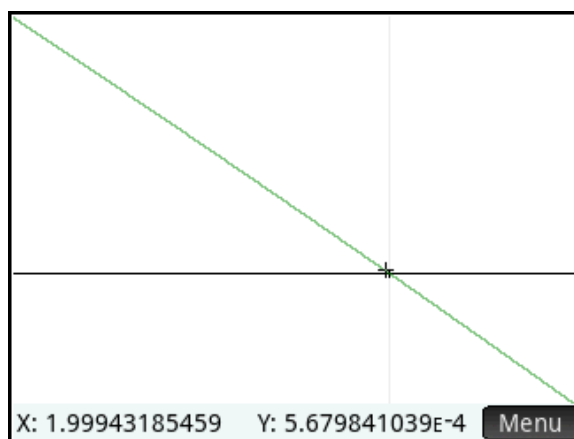
Move the tracer to the point (2,0) and press  $\left[ \begin{smallmatrix} + \\ \text{Ans} \end{smallmatrix} \right]$  to zoom in on this point. Drag the display with your finger to scroll the window. Continue to zoom and scroll until the plot fills the window as in the figure to the right.

Zoom in on the point (2,0) using the open pinch gesture. Place two fingers on the display at the same time and move them apart to zoom in. A vertical or horizontal pinch will zoom in one dimension only, while any diagonal pinch will zoom in both dimensions. Zoom in to establish linearity and then zoom vertically or horizontally to get the line to pass through two opposite corners of the display.

We will now use the window dimensions to estimate the slope of the tangent line. Press  $\left[ \begin{smallmatrix} \text{Home} \\ \text{Settings} \end{smallmatrix} \right]$  to open the Home view. Press  $\left[ \begin{smallmatrix} \text{Units} \\ \text{Units} \end{smallmatrix} \right]$  and choose the fraction template. Press  $\left[ \begin{smallmatrix} \text{Vars} \\ \text{Chans} \end{smallmatrix} \right]$ , tap  $\left[ \begin{smallmatrix} \text{App} \\ \text{App} \end{smallmatrix} \right]$ , tap Advanced Graphing, tap Plot, and select Ymax. Continue in this fashion to build the expression  $\frac{Y_{\text{max}} - Y_{\text{min}}}{X_{\text{min}} - X_{\text{max}}}$ . Press  $\left[ \begin{smallmatrix} \text{Enter} \\ \text{=} \end{smallmatrix} \right]$  to evaluate. Our estimate for the slope is close to -1.

We will now use implicit differentiation to find the slope exactly. Press  $\left[ \begin{smallmatrix} \text{CAS} \\ \text{Settings} \end{smallmatrix} \right]$  to open the CAS view. Press  $\left[ \begin{smallmatrix} \text{Menu} \\ \text{Menu} \end{smallmatrix} \right]$  to open the Toolbox Menus, tap  $\left[ \begin{smallmatrix} \text{Catlg} \\ \text{Catlg} \end{smallmatrix} \right]$ , then press  $\left[ \begin{smallmatrix} \text{TAN} \\ \text{ATAN} \end{smallmatrix} \right]$   $\left[ \begin{smallmatrix} +/\text{--} \\ \text{IM} \end{smallmatrix} \right]$  (IM) to jump to commands that start with those two letters. Scroll down to `implicit_diff`. Press  $\left[ \begin{smallmatrix} \text{Help} \\ \text{User} \end{smallmatrix} \right]$  to view the help page for this command. Notice the menu keys:

- **Tree**: opens the entire help tree
- **Exempl**: opens a menu of examples to paste into the CAS
- **Page**: page by page navigation
- **Other**: view related commands
- **OK**: close the help page



We can see that the command takes an expression, followed by the variables for differentiation.

Tap **OK** to close the help page. Tap **OK** to paste the command into the CAS. The CAS uses lower-case variable names, so use  $x$  and  $y$  here.

Enter the expression for our ellipse, followed by both  $x$  and  $y$ , and press **Enter** to see the result. Tap **simplif** to simplify the expression.

CAS Advanced Graphing 10:58

$$\text{implicit\_diff}(x^2 + 2xy + 3y^2 - 4, x, y)$$

$$\frac{-2x - 2y}{2x + 6y}$$

$$\text{simplify}\left(\frac{-2x - 2y}{2x + 6y}\right)$$

$$\frac{-x - y}{x + 3y}$$

Sto ► simplif

Press **Symb** and select the third template in the first row (called the where() command). Tap on the first square, then tap on our last result and tap **Copy** to copy it into the square. Tap on the second square and enter  $x=2, y=0$ . Press **Enter** to see the result: the slope is  $-1$ .

CAS Advanced Graphing 10:57

$$\text{implicit\_diff}(x^2 + 2xy + 3y^2 - 4, x, y)$$

$$\frac{-2x - 2y}{2x + 6y}$$

$$\text{simplify}\left(\frac{-2x - 2y}{2x + 6y}\right)$$

$$\frac{-x - y}{x + 3y}$$

$$\left[\frac{-x - y}{x + 3y}\right]_{[x=2 \ y=0]}$$

$$-1$$

Sto ► simplif

Press **Symb** to return to Symbolic view and enter the equation of the tangent line. The line whose slope is  $-1$  and contains  $(2,0)$  is  $y - 0 = -1(x - 2)$  or  $y = -x + 2$ . Enter this equation in V2 and press **Plot** to see the graph. You can enter the equation in either point-slope form or slope-intercept form. Press **Base** to zoom back out so you can see the entire ellipse.



## Extension

Find any values of  $a$  for which the parabola  $x = a \cdot (y^2 + 5)$  is also tangent to our line  $y = -x + 2$ .

Press **CAS** to return to the CAS. Differentiate our equation implicitly with respect to both  $x$  and  $y$ , as shown in the figure to the right.

Press **Mem**, tap **CAS**, tap **Solve**, and select **solve**. Now copy the expression (just double-tap it) and set it equal to -1 as shown. Press **Eval**, followed by  $y$  to solve for  $y$  and press **Enter** to see the result.

CAS Advanced Graphing 10:55

|  |   |
|--|---|
| $2*x+6*y$  |   |
| $\text{simplify}\left(\frac{-2*x-2*y}{2*x+6*y}\right)$ | $\frac{-x-y}{x+3*y}$                                  |
| $\frac{-x-y}{x+3*y}$                                   | $\left[\begin{matrix} x=2 \\ y=0 \end{matrix}\right]$ |
|  | -1  |
| $\text{implicit\_diff}(x=a*(y^2+5),x,y)$               | $\frac{1}{2*a*y}$                                     |
| $\text{solve}\left(\frac{1}{2*a*y}=-1,y\right)$        | $\left[\frac{1}{2*a}\right]$                          |

Sto ► simplif

We know that the point of tangency satisfies both equations (the parabola and the line). Return to the **where()** command and enter the parabola equation as shown in the figure to the right. In the second square enter the substitution  $x = 2 - y$  to get an equation in  $a$  and  $y$ .

Repeat with the result and our substitution for  $y$  in terms of  $a$ , given by our implicit derivative equation.

CAS Advanced Graphing 10:54

|   |  |
|---|--|
| $\text{solve}\left(\frac{1}{2*a*y}=-1,y\right)$ | $\left[\frac{1}{2*a}\right]$                                     |
| $x=a*(y^2+5)$                                   | $x=2-y$  |
|   | $-y+2=a*(y^2+5)$   |
| $-y+2=a*(y^2+5)$                                | $y=\left[\frac{-1}{2*a}\right]$                                  |
|   | $2+\frac{1}{2*a}=a*\left(\left(\frac{-1}{2*a}\right)^2+5\right)$ |

Sto ► simplif

Now solve for  $a$ . The result shows that there are two values of  $a$ ,  $-1/10$  and  $1/2$ , for which the graph of  $x = a \cdot (y^2 + 5)$  is tangent to the line  $y = -x + 2$ .

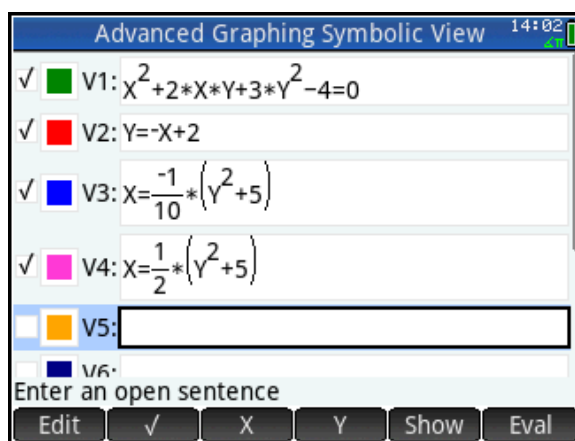
CAS Advanced Graphing 10:53

|   |  |
|---|--|
| $x=a*(y^2+5)$   | $x=2-y$  |
| $-y+2=a*(y^2+5)$  | $y=\left[\frac{-1}{2*a}\right]$                                  |
|   | $2+\frac{1}{2*a}=a*\left(\left(\frac{-1}{2*a}\right)^2+5\right)$ |
| $\text{solve}\left(2+\frac{1}{2*a}=a*\left(\left(\frac{-1}{2*a}\right)^2+5\right),a\right)$ | $\left[\frac{-1}{10}, \frac{1}{2}\right]$                        |

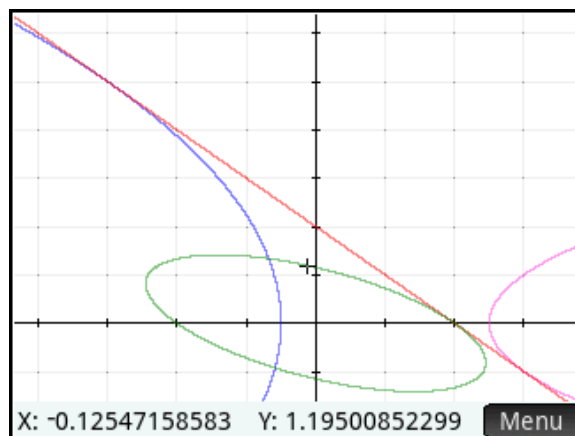
Sto ► simplif



Return to Symbolic view and enter these equations in V3 and V4.



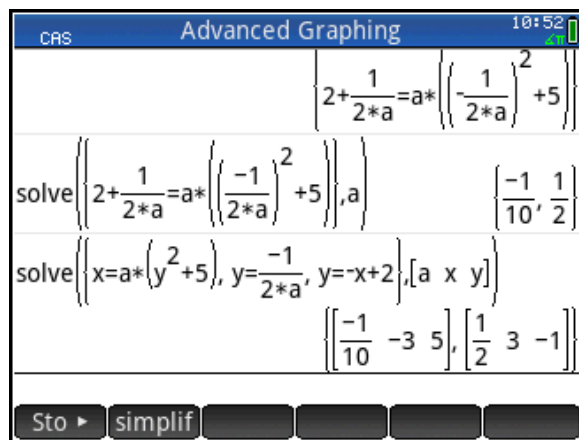
Press to see the graphs.



You can perform the entire process in one step if you like. Use the solve() command with the set of equations enclosed in curly braces as a system, followed by the vector of variables. The figure to the right shows the exact expression and the result. The result shows that:

- When  $a = -1/10$ , the parabola is tangent to the line at the point  $(-3, 5)$
- When  $a = 1/2$ , the parabola is tangent to the line at the point  $(3, -1)$

These results can be verified in Plot view.



The HP Prime CAS gives you extraordinary flexibility in exploring and solving problems exactly.

# Approximating Integrals with Riemann Sums


## HP Prime Functionality Introduced:

Using a custom **View** menu to navigate a user App.

## AP Calculus Content:


Introduction to Riemann Sums using area; convergence of Riemann Sums and behavior of their error

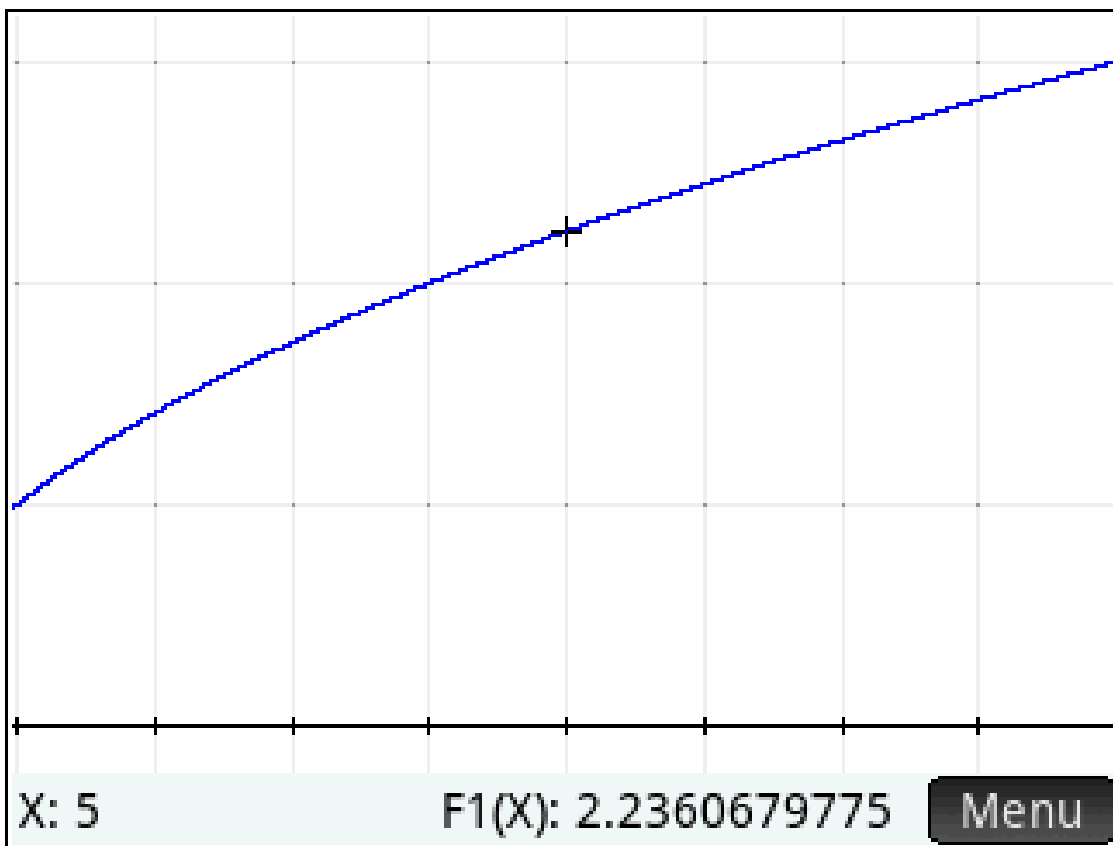
## Activity:



Students connect to the HP Classroom Network, and the teacher distributes the NumInt App. Press , and select NumInt. The App is based on the built-in Function App.


In this activity, you will work with the function  $f(x) = \sqrt{x}$ . In particular, you will approximate the definite integral,  $\int_1^9 \sqrt{x} dx$ , which represents the area underneath the graph of  $f(x)$  from  $x = 1$  to  $x = 9$ .

The first step in approximating an integral is to sub-divide your interval. In this example, you're working with the interval from  $x = 1$  to  $x = 9$ .

1. Press  and look at the graph of  $F1(X) = \sqrt{X}$ .



First, you will use 2 subintervals,  $[1,5]$  and  $[5,9]$ . You'll evaluate your integrand function,  $F1(X) = \sqrt{X}$ , at the left hand endpoint of each of these intervals,  $X = 1$  and  $X = 5$ , using the Numeric View to see the results. Press   (Num Setup). Notice that the table setup (**Num Type**) is set to Build Your Own table. Enter 1 and 5 into the column for  $X$ .

2. On your graph in number 1, draw the rectangles with width 4 and heights given by the values of  $F1(1)$  and  $F1(5)$ . To do this, place your pencil on the  $x$ -axis at  $X = 1$ . Draw a vertical line segment up to the graph of  $F1(X)$ . The length of that line segment is  $F1(1) = \sqrt{1} = 1$ . Then draw a horizontal line segment from the point  $(1, 1)$  straight across to the point  $(5, 1)$ . Place your pencil on the  $x$ -axis at  $X = 5$ , and repeat the process of drawing a vertical line up to the graph of  $F1(X)$ , then a horizontal line to the point  $(9, \sqrt{5})$ . Complete the second rectangle by drawing a vertical segment down to the  $x$ -axis at  $X = 9$ .
3. Press . Write out a calculation in the form of the sum of two products that gives the sum of the areas of the two rectangles you drew in question 2. Evaluate the sum, and write down its value, accurate to three digits to the right of the decimal point.

---

This sum is called the Left-Hand Riemann Sum with 2 subintervals for  $F1(X)$  on the interval  $[1,9]$ . Look at the rectangles you drew on your graph.

4. Is the value of your left-hand sum smaller or larger than the exact area under the graph of  $F1(X)$  from 1 to 9? Explain your answer.

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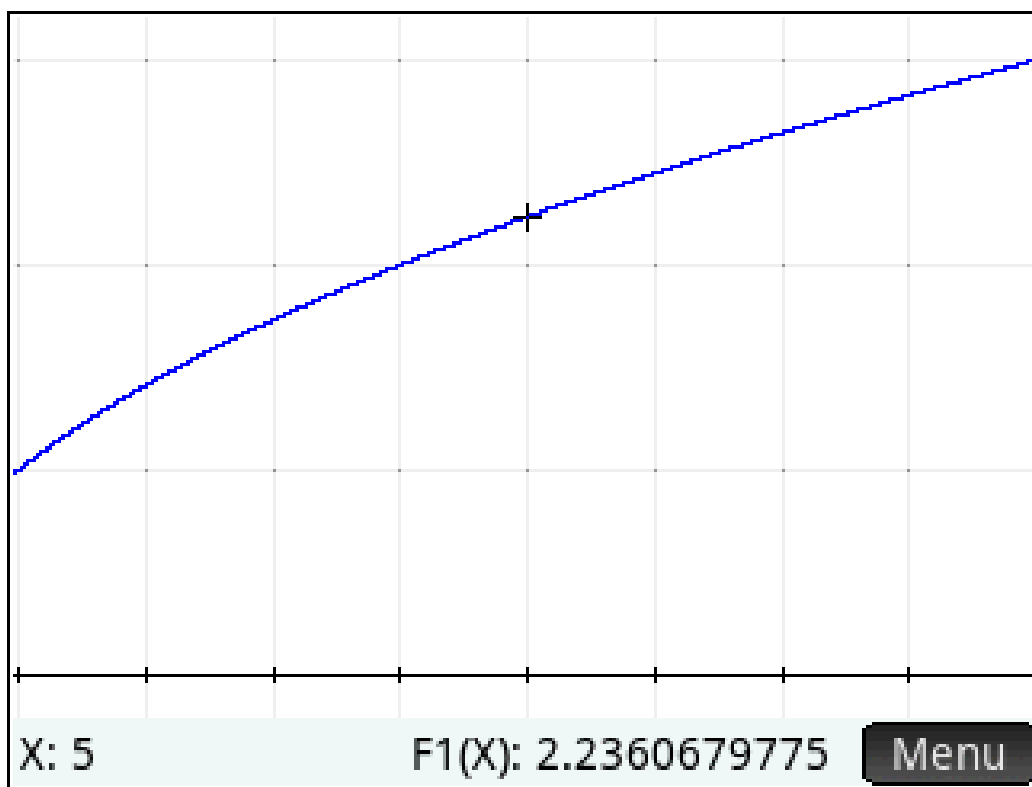
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To improve your approximation of the area, all you need to do is include more sub-intervals so that the width of each rectangle is smaller. First, you will use 4 rectangles, each of width 2.

5. Press **Shift** **Num** **→ Setup**, and change NumStep to 2.

6. On the graph, draw the 4 rectangles with width 2 and heights given by the values of  $F1(X)$  from your table, following the procedure outlined in question 2.



7. Write out a calculation in the form of the sum of four products that gives the sum of the areas of the four rectangles you drew in question 7. Also calculate the sum, accurate to three decimal places.

---






8. Is the sum larger or smaller than the exact area under the curve? \_\_\_\_\_

9. Is the sum larger or smaller than the sum you calculated using only two rectangles?

---

Draw a picture that supports your answer to question 9.

Generally, by taking more rectangles in the sum, you get better and better approximations. However, these calculations soon become tedious. The NumInt App will help you automate the calculations.

Press . Select **5 Limits** and enter 1 for the lower limit and 9 for the upper limit (these may already be pre-set to the correct values). Select **4 Select Method**, and **1 LEFT** to do left hand Riemann Sums. Then select **3 Enter num subintervals**. Enter 2 for the number of subintervals and press . Then select **6 Plot** and you will see the two rectangles drawn and the sum computed. Compare this with your answers to questions 2 and 3. Then press  and . Repeat this process to draw 4 rectangles and compare the results with your answers to questions 6 and 7. Then press  and select **7 Numeric Results**. This takes you to the numeric view where you can enter the number of subintervals in the X column of the table. The value of the sum is computed for the currently selected method, left or right. Enter 4 for X, and you will also see the sum displayed for 4 left hand rectangles. Use the numeric view to complete the table below for Left Hand Riemann Sums.

10.

| Number of Subintervals | Left Hand Riemann Sum | Right Hand Riemann Sum |
|------------------------|-----------------------|------------------------|
| 8                      |                       |                        |
| 16                     |                       |                        |
| 32                     |                       |                        |
| 64                     |                       |                        |
| 128                    |                       |                        |
| 256                    |                       |                        |
| 512                    |                       |                        |
| 1024                   |                       |                        |
| 2048                   |                       |                        |

11. Does it appear the values of the Left Hand Riemann Sums approach a limiting value as the number of subintervals gets larger? \_\_\_\_\_ If so, what is that value?

\_\_\_\_\_

On the View menu, Select **4 Select Method**, and **3 RIGHT** to do right hand Riemann Sums. Fill in the right hand column in the table as before.

12. Are the Right Hand Riemann Sums larger or smaller than the exact area under the curve?

\_\_\_\_\_

13. Explain why the Right Hand Riemann Sums decrease as the number of rectangles increases.

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14. Does it appear the values of the Right Hand Riemann Sums approach a limiting value as the number of subintervals gets larger? \_\_\_\_\_ If so, what is that value?

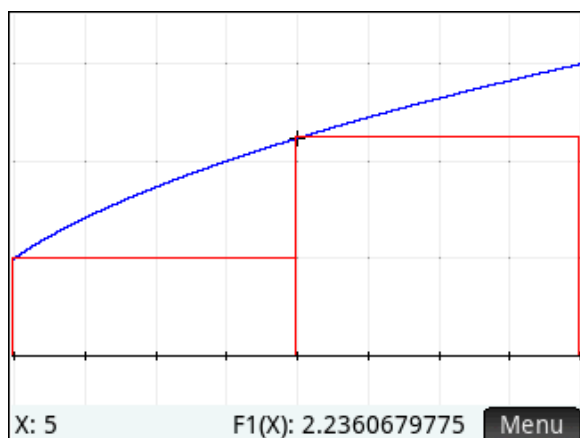
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# Answers

1.

| NumInt Numeric View  |               | 13:26 |
|----------------------|---------------|-------|
| X                    | F1            |       |
| 1                    | 1             |       |
| 5                    | 2.2360679775  |       |
| 9                    | 3             |       |
| 13                   | 3.60555127546 |       |
| 17                   | 4.12310562562 |       |
| 21                   | 4.58257569496 |       |
| 25                   | 5             |       |
| 29                   | 5.38516480713 |       |
| 33                   | 5.74456264654 |       |
| 37                   | 6.0827625303  |       |
| 1                    |               |       |
| Zoom More Go To Defn |               |       |

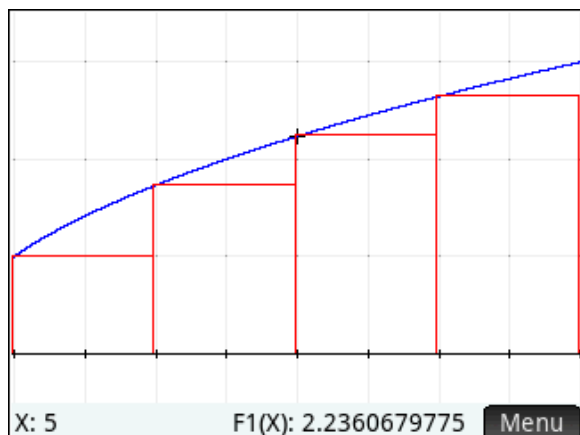
2.



3.  $4 \cdot 1 + 4 \cdot \sqrt{5} \approx 12.944$

4. This is smaller than the exact area under the graph. Each rectangle leaves out some of the region under the graph of F1(X).

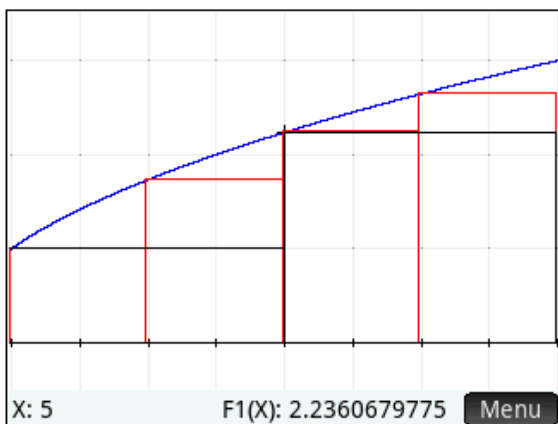
6.



7.  $2 \cdot 1 + 2 \cdot 1.73205 + 2 \cdot 2.23607 + 2 \cdot 2.64576 \approx 15.228$

8. Smaller

9. The sum of areas of the 4 rectangles is larger than the sum of the areas of two rectangles. The picture below supports this conclusion.



10.

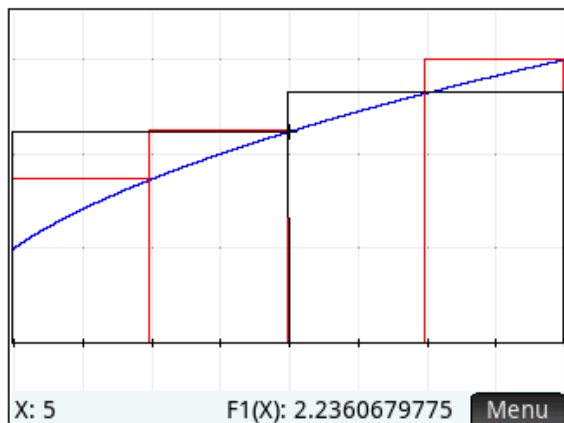
| Number of Subintervals | Left Hand Riemann Sum | Right Hand Riemann Sum |
|------------------------|-----------------------|------------------------|
| 8                      | 16.306                | 18.306                 |
| 16                     | 16.826                | 17.826                 |
| 32                     | 17.082                | 17.582                 |
| 64                     | 17.208                | 17.458                 |
| 128                    | 17.271                | 17.396                 |
| 256                    | 17.302                | 17.365                 |
| 512                    | 17.318                | 17.349                 |
| 1024                   | 17.326                | 17.341                 |
| 2048                   | 17.329                | 17.337                 |

11. Yes, 17.333

12. The Right Hand Riemann Sums are larger than the area under the curve, because the integrand is increasing.



13. As we increase the number of subintervals, the right hand rectangles include less and less "extra" area. The picture below supports this conclusion.



14. Yes; 17.333

### Extensions:

- E1) Repeat the activity using the midpoint rule. Like the activity, have students first produce a table of inputs and outputs (if 2 subintervals are used over the interval  $x = 1$  to  $x = 5$ , the inputs should be  $x = 2$  and  $x = 4$ ). UseTblStart = 2 and  $\Delta\text{Tbl} = 2$ . Then change  $\Delta\text{Tbl}$  to 0.5. Finally, use the calculator program and just add an extra column to the table from question 13.
- E2) Investigate a function that is decreasing over the interval of integration, such as  $\int_0^{\pi} \cos(x) dx$ . Ask students to predict which method, left or right hand sum, will over-approximate the integral. Alternately, ask students to produce a function  $g$  for which the left hand sum will under-approximate  $\int_1^5 g(x) dx$ .
- E3) Repeat the activity using the trapezoid rule. Demonstrate that the trapezoid rule is equivalent to averaging the left and right hand sums. The NUMINT program leaves in its wake the most recently calculated sums in the list variable, L6. L6(1) is the left hand sum, L6(2) the right hand sum, L6(3) the midpoint sum, and L6(4) the trapezoid sum. You could calculate  $(L6(1) + L6(2))/2$  and compare the result to L6(4), for example.
- E4) Sometime after students know how to evaluate a definite integral using the Fundamental Theorem of Calculus (so that the exact value of an integral is known), return to the NUMINT program, and use the values in L6 to explore the errors from using the various approximations.

# Fundamental Theorem Investigation

## HP Prime Functionality Introduced:

Defining a function in terms of an integral, the MAKELIST() and other List commands,

## AP Calculus Content:


### The Fundamental Theorem of Calculus

### Activity:




In this activity, students explore the Fundamental Theorem of Calculus from numerical and graphical perspectives. The exploration gives students additional practice with functions of the form

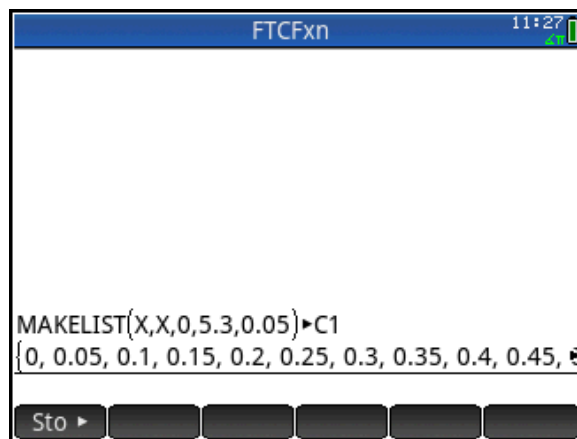
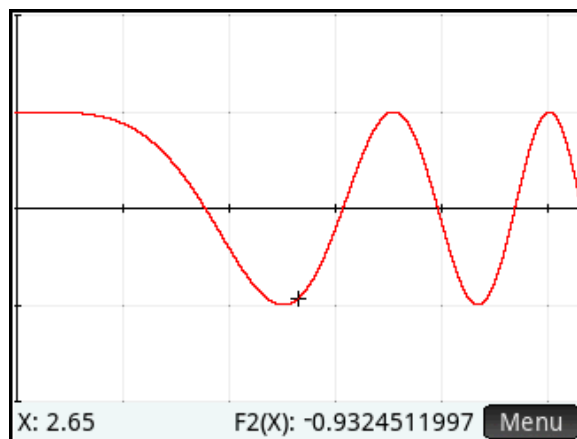
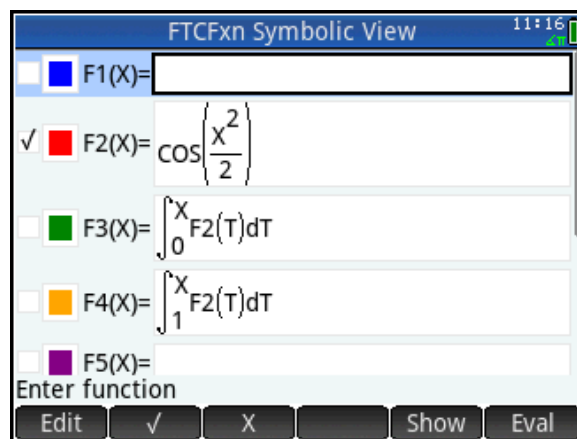
$$F(x) = \int_0^x f(t) dt.$$

Press  and select the FTCFxn App. Study the function definitions in the Symbolic View.

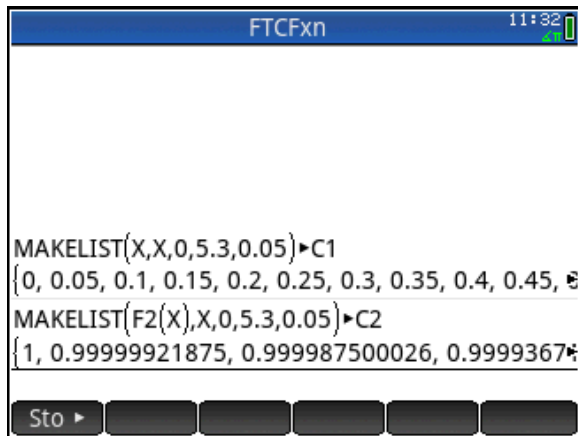
Notice the F3(X) and F4(X) are functions defined by a definite integral. That is, the independent variable of the functions is the upper limit of integration. Also notice that that F2(X) is the only function selected for graphing. Press  and look at the graph. Your graph should look like the one shown.

Now, you will create three lists. The first list, **C1**, consists of values of the independent variable, **X**, from the graph screen. These will serve as inputs into F3(X). To create the lists, follow these steps:

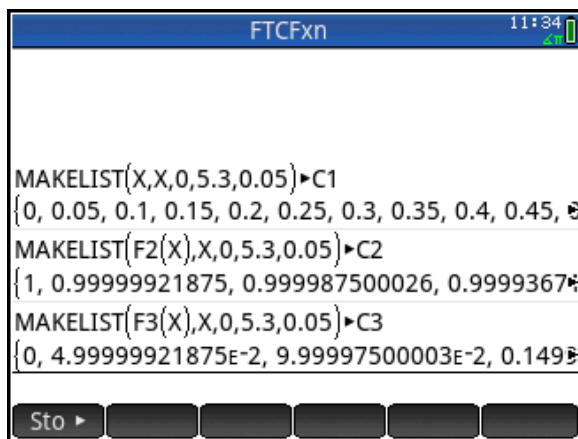
- Press  and the **Math** menu key, then select **MAKELIST** from the **LIST** group.
- Fill in the arguments as shown at right.
- Press the **Sto** menu key.
- Press  , then enter 1 to enter C1.



Now you will evaluate  $F2(X)$  at each of the inputs in  $C1$ , and store the results in another list,  $C2$ . Touch the **MAKELIST** command that you just entered with your finger to select it, and then press **Copy**. Navigate to the first argument,  $X$ , and replace it with  $F2(X)$ . Then navigate to the end of the edit line and replace  $C1$  with  $C2$ . Press **Enter** to generate and store the list of outputs from  $F2(X)$  into the  $C2$ .



Repeat this procedure to evaluate  $F3(X)$  at each of the inputs and store the results in list  $C3$  as shown.



Remember that each element of  $C3$  results from the evaluation of a definite integral. For example, when 0.05, the second value of  $C1$  is plugged into  $F3(X)$ , the value of

$$\int_0^{0.05} \cos\left(\frac{T^2}{2}\right) dT = 0.04999992$$

results, which is the second value in  $C3$ .

1. Explain why the first value in  $C3$ , which results from evaluating  $F3(0)$ , is 0.

Press **Plot** and look at the graph of the integrand,  $F2(X)$ . Use the graph to answer questions 2 and 3.

2. Why is the second value in  $C3$  greater than 0?

3. Why is the third value in C3, which represents  $F3(0.1) = \int_0^{0.1} \cos\left(\frac{T^2}{2}\right) dT$ , greater than the second value in C3, which represents  $F3(0.05) = \int_0^{0.05} \cos\left(\frac{T^2}{2}\right) dT$  ? \_\_\_\_\_
- 
- 

Press **Apps** and select **Statistics 2Var** to see a table of inputs and outputs. Remember that C1 contains the  $X$  values; C2 contains the values of the integrand,  $\cos\left(\frac{T^2}{2}\right)$ ; and C3 contains the values of the integral,  $\int_0^X \cos\left(\frac{T^2}{2}\right) dT$ . Use the cursor keys or swipe with your finger to navigate down the C3 column.

4. From the list of data values stored in C3 for what value of  $X$  (stored in C1) does

$$F3(X) = \int_0^X \cos\left(\frac{T^2}{2}\right) dT \text{ appear to reach its first local maximum?}$$


---

5. Explain why the 38<sup>th</sup> value is smaller than the 37<sup>th</sup> value in C3. That is, why is

$$\int_0^{1.85} \cos\left(\frac{T^2}{2}\right) dT < \int_0^{1.8} \cos\left(\frac{T^2}{2}\right) dT ?$$


---



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6. From the list of data values stored in C3, for what positive value of  $X$  does

$$F3(X) = \int_0^X \cos\left(\frac{T^2}{2}\right) dT \text{ appear to reach its first local minimum?}$$


---

Press **Apps** and select **FTCFxn** to return to the FTCFxn App, and look at the plot of the integrand again. Press **Menu**, **Fcn**, and select **2 Root**. Move the cursor close to the smallest x-intercept on the graph of  $F2(X)$ .

7. Record the x-intercept. \_\_\_\_\_

8. Move the cursor close to the next largest x-intercept, and use the **Root** command to find it.


Record the next largest x-intercept. \_\_\_\_\_

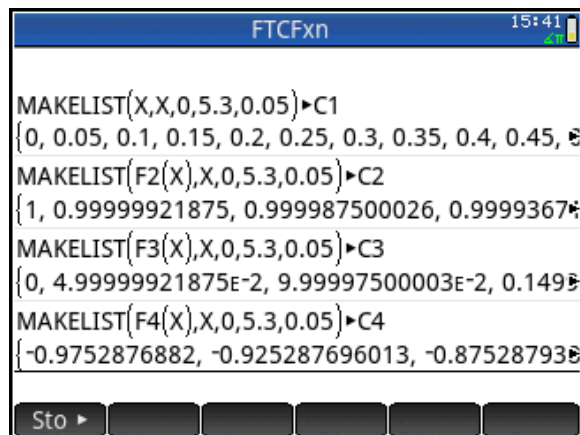
Compare these Roots to your answers to numbers 4 and 6.



9. Find the next positive  $X$ -intercept of the graph of  $F2(X)$  (close to  $X = 4$ ). \_\_\_\_\_

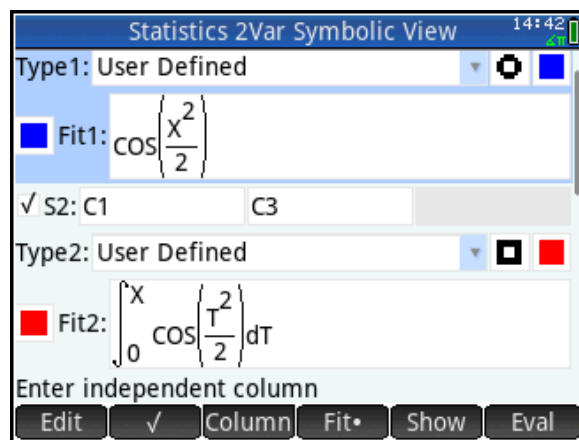
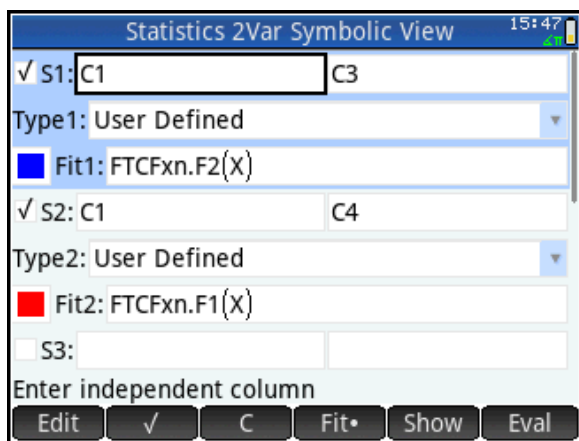
10. What happens with  $F3(X)$  at the point found in question 9? \_\_\_\_\_

Verify your answer numerically by going back the Statistics2Var App and looking at the table of values for  $F3(X)$  in C3.

Press , copy the most recent **MAKELIST** command to the edit line, and change the arguments as shown to evaluate  $F4(X)$  at all the inputs in C1 and store the results in C4.



Now you'll create scatter plots of C3 versus C1 and C4 versus C1. Press  and select **Statistics 2Var**. Press  to start the Symbolic View. Enter the setups shown in the screen at right.



Then press **Shift** **Plot Setup** to go to the Plot Setup View.  
Enter the settings shown.

Statistics 2Var Plot Setup 15:47

S1 Mark: S2 Mark: S3 Mark:

S4 Mark: S5 Mark:

X Rng: 0 5.3

Y Rng: -2 2

X Tick: 1

Y Tick: 1

Statistics 2Var Plot Setup 14:44

X Rng: 0 5.3

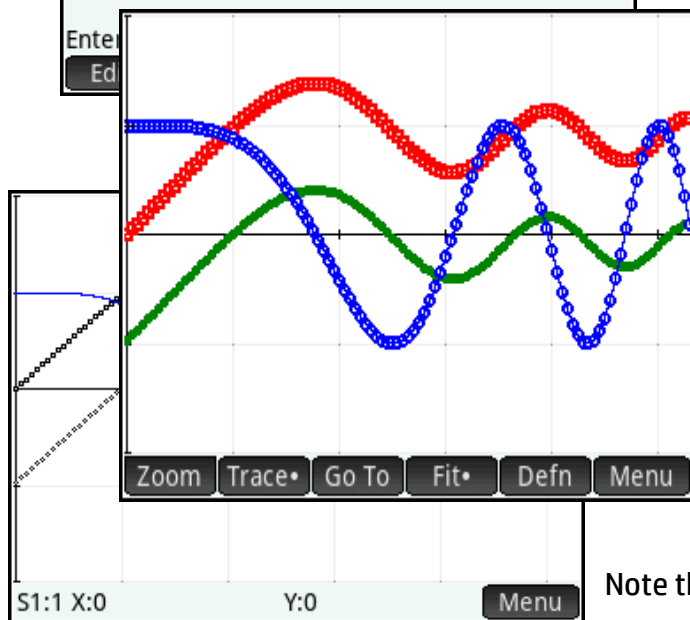
Y Rng: -2 2

X Tick: 1

Y Tick: 1

Enter minimum horizontal value

Edit Page 1/2



Press **Plot Setup** to see the graph.

Note that the first element of C4 represents

$$F4(0) = \int_1^0 \cos\left(\frac{T^2}{2}\right) dT = -\int_0^1 \cos\left(\frac{T^2}{2}\right) dT.$$

11. Why is the graph of C4 versus C1 (in scatter plot S2) below the graph of C3 versus C1 (in scatter plot S1)? (Remember, the only difference between the definitions of F3 and F4 is the lower limit of integration.)

12. It appears that corresponding values in C4 and C3 differ by a constant, i.e. that  $F4(X) - F3(X)$  is the same for all values of X. Find the value of this constant.


(You could calculate this in several different ways. If you get stuck, try expressing  $F4(X) - F3(X)$  in terms of an integral.)

13. Explain why plots S1 and S2 have the same shape.

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Press  and look at the table of values for C3 and C4, and notice that the locations of the extrema are the same for both functions (as you would expect from an inspection of the scatter plots).

14. Look at the locations of the local minima and the local maximum on the graph of F2(X). At these values of X, describe what appears to happen with the concavity of the graphs from S1 and S2.

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15. Write an equation with a derivative on one side that shows how F2(X) and F3(X) are related.

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16. Recall that the derivative of a function  $f$  at  $x = a$  can be defined by

$$f'(a) = \lim_{x \rightarrow a} \frac{f(x) - f(a)}{x - a}.$$

Approximate  $F3'(1)$  by evaluating the expression  $(F3(1.1) - F3(1))/0.1$ .  
\_\_\_\_\_. Your answer should be close to the value of  $F2(1)$ . Why?

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17. Approximate  $F4'(1)$  by evaluating the expression  $(F4(1.1) - F4(1))/0.1$ . \_\_\_\_\_.  
Again, your answer should be close to the value of  $F2(1)$ . Why?

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
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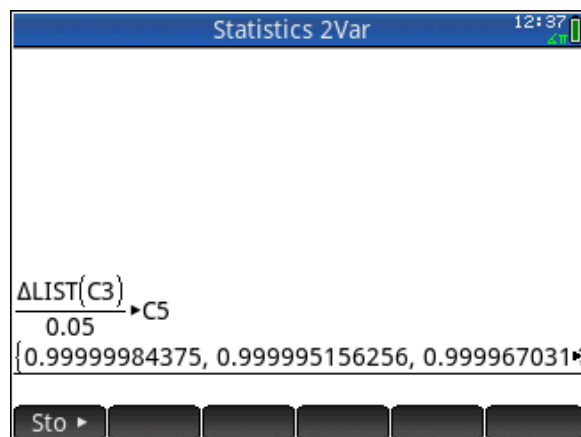
## Extension

This extension is a must-do!

After students have seen that  $\frac{d}{dX} F3(X) = F2(X)$ , or  $\frac{d}{dX} \int_0^X \cos\left(\frac{T^2}{2}\right) dT = \cos\left(\frac{X^2}{2}\right)$ , it is natural

to gather more evidence in support of this conclusion. Because we have assembled data of inputs and outputs for **F3**, and we have a formula for **F2**, it is a relatively simple matter to form difference quotients for **F3** and compare them to outputs from **F2**. This was done for a single value of **X** in question 16. Here's one way to do it for all the values of **F3** that we've calculated.

You will form the difference quotients using the calculator's  $\Delta$ List command. That command generates a new list, with each entry formed by subtracting adjacent pairs of the argument. On the **Home** screen, press  **Math** 6 1 to select the  $\Delta$ List command. Enter the command  $\Delta\text{List}(\text{C3})/0.05$   **C5**. Applying the  $\Delta$ List command to the list, **C3**, and then dividing that list by the differences in the input list, 0.05, gives a list of difference quotients.



The list C5 contains one entry fewer than C1, C2, C3, and C4. So, you need to manually calculate and enter the missing value at the end of C5. Note that it should represent the average rate of change of F3 on the interval from  $X = 5.3$  to  $X = 5.35$ . So it is

|     | C2         | C3         | C4         | C5        |
|-----|------------|------------|------------|-----------|
| 99  | 0.84652621 | 0.77109531 | -0.2041923 | 0.9031231 |
| 100 | 0.9507590  | 0.8162515  | -0.1590361 | 0.9792942 |
| 101 | 0.9977982  | 0.8652162  | -0.1100714 | 0.9956336 |
| 102 | 0.9829584  | 0.9149979  | -6.0289774 | 0.9493091 |
| 103 | 0.9053346  | 0.9624633  | -1.2824314 | 0.8414532 |
| 104 | 0.7681309  | 1.0045360  | 2.9248349  | 0.6773642 |
| 105 | 0.5787270  | 1.0384042  | 6.3116562  | 0.4664271 |

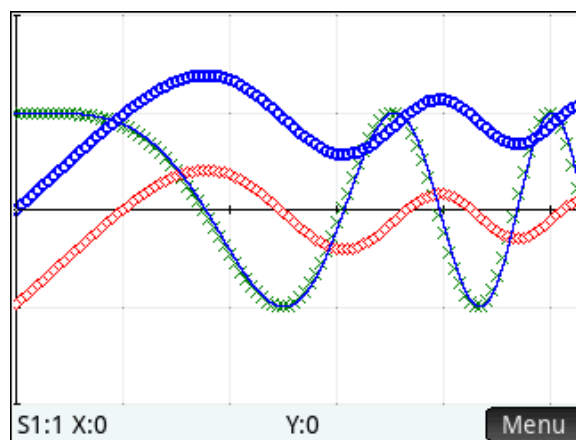
Below the table, the manual calculation is shown:

$$\frac{\int_{5.3}^{5.35} \cos\left(\frac{T^2}{2}\right) dT}{0.05}$$

$$\frac{\left(\int_0^{5.35} \cos\left(\frac{T^2}{2}\right) dT - \int_0^{5.30} \cos\left(\frac{T^2}{2}\right) dT\right)}{0.05} = \frac{\left(\int_{5.30}^{5.35} \cos\left(\frac{T^2}{2}\right) dT\right)}{0.05}$$
. Position the cursor in the cell for C5(107) and enter that expression, as shown.



Then set up a scatter plot of C5 versus C1, and behold!



## Answers

1. The first value in **C3** represents  $\int_0^0 Y2(T)dT$  and so is zero. There is no area from 0 to 0.
2. The second value in **C3** is positive since the integrand is positive for all  $T$  in the interval  $[0, 0.1]$ .
3. The third value in **C3** is greater than the second since the integrand is positive for all  $x$  in the interval  $[0.1, 0.2]$ .
4. 1.8
5.  $\int_0^{1.9} \cos\left(\frac{T^2}{2}\right) dT < \int_0^{1.8} \cos\left(\frac{T^2}{2}\right) dT$  because the integrand is negative for all  $T$  in the interval  $[1.8, 1.9]$ . Thus, the function is decreasing.
6. 3.1

In questions 7-10, students should observe that the function **F3** has a maximum where the integrand changes sign from positive to negative, and a minimum where the integrand changes sign from negative to positive. Again, this connection is meant to recall students' experience with the locations of maximum and minimum points and their relationship with the sign of the derivative. A question like, "Have you ever seen a situation in which one 'thing' reaches a maximum where another 'thing' changes sign from positive to negative? What are the two 'things'?" Students make the connection explicit in question 15.

7. (A) 1.77245 and (B) 3.06998
8. At the first  $x$ -intercept, 1.77245, the integrand changes sign from positive to negative. Thus the function, **F3**, has a local maximum there. We stop accumulating positive values of **F2**, and start accumulating negative values at that point. Similarly, at  $x = 3.06998$ , the integrand changes sign from negative to positive. Thus, we stop accumulating negative values and start accumulating positive values at that point, producing a local minimum on the graph of **F3**.
9. 3.963327
10. **F3** has a maximum there. Looking at the table of values for **C1**, **C2**, and **C3**, this occurs near **X** = 4.
11. Answers will vary. Focusing on **X** = 0,  $F3(0) = \int_0^0 \cos\left(\frac{T^2}{2}\right) dT$  is zero, since the upper and lower limits of integration are equal. However,  $F4(0) = \int_1^0 \cos\left(\frac{T^2}{2}\right) dT = -\int_0^1 \cos\left(\frac{T^2}{2}\right) dT < 0$ , since the integral is positive. This puts the graph of **C4** below the graph of **C3**. Also see the answer to question 12.

12. The difference is a constant. Just calculate  $\mathbf{F4(0)} - \mathbf{F3(0)} = -0.975288$ . Students may notice that

$$\mathbf{F4(X)} - \mathbf{F3(X)} = \int_1^X \cos\left(\frac{T^2}{2}\right) dT - \int_0^X \cos\left(\frac{T^2}{2}\right) dT = -\int_0^1 \cos\left(\frac{T^2}{2}\right) dT \text{ and simply evaluate the}$$

latter integral, or note that it is the negative of  $\mathbf{C4(1)}$ .

The punch line is at hand in question 13! The Fundamental Theorem says that the two functions,  $\mathbf{F3}$  and  $\mathbf{F4}$ , have the same derivative,  $\mathbf{F2}$ . The derivative of a function tells you the slope on the graph of the function, and graphs that have the same slope everywhere have the same shape. In fact,  $\mathbf{S1}$  and  $\mathbf{S2}$  are members of the family of antiderivatives of  $\mathbf{F2}$ . Two antiderivatives of a function always differ by a constant.

13.  $\mathbf{S1}$  and  $\mathbf{S2}$  have the same shape since each accumulates the same amount of area from the graph of  $\mathbf{F2}$  as  $\mathbf{X}$  increases. So each increases at the same rate where the integrand is positive, and each decreases at the same rate where the integrand is negative.

If you have addressed the connection between points of inflection and extrema of the first derivative (in addition to the more customary definition as points where the second derivative changes sign), then question 14 should give students yet another indicator that  $\mathbf{F2}$  is indeed the derivative of  $\mathbf{F3}$ .

14. The graphs of  $\mathbf{S1}$  and  $\mathbf{S2}$  both appear to have a change of concavity at the points where the graph of  $\mathbf{F2}$  has a local maximum or a local minimum. The points of inflection occur at  $\mathbf{X} = 2.506629$ ,  $\mathbf{X} = 3.544908$ , and  $\mathbf{X} = 4.341608$ . These are the extrema on the graph of  $\mathbf{F2}$ .

15.  $\frac{d}{dX} F3(X) = F2(X)$  or better stated,  $\frac{d}{dX} \int_0^X \cos\left(\frac{T^2}{2}\right) dT = \cos\left(\frac{X^2}{2}\right)$ .

Questions 16 and 17 lead students to approximate the derivative of  $\mathbf{F3}$  by forming difference quotients. The answers are not exactly the same as the output from  $\mathbf{F2}$ . To get better agreement, you could have students form a difference quotient with a smaller step size as follows:

Evaluate  $\frac{F3(1.0001) - F3(1)}{0.0001}$ , and compare it to  $\mathbf{F2(1)}$ . They agree to four digits to the right of the decimal!

16.  $(\mathbf{F3(1.1)} - \mathbf{F3(1)})/0.1 = 0.851261$ .  $\mathbf{F2(1)} = 0.877583$

17.  $(\mathbf{F4(1.1)} - \mathbf{F4(1)})/0.1 = 0.851261$ .  $\mathbf{F2(1)} = 0.877583$

Notice that the difference quotients are identical.  $\mathbf{F3}$  and  $\mathbf{F4}$  are changing by exactly the same amount, an amount dictated by the magnitude of  $\mathbf{F2}$ .

# Differential Equations



## HP Prime Functionality Introduced:

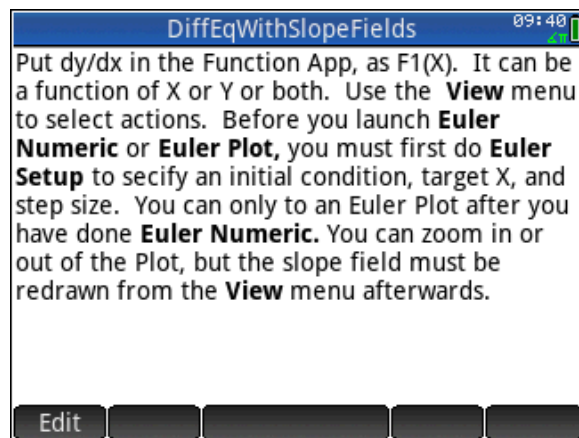
Using CAS to solve a differential equation




## AP Calculus Content:

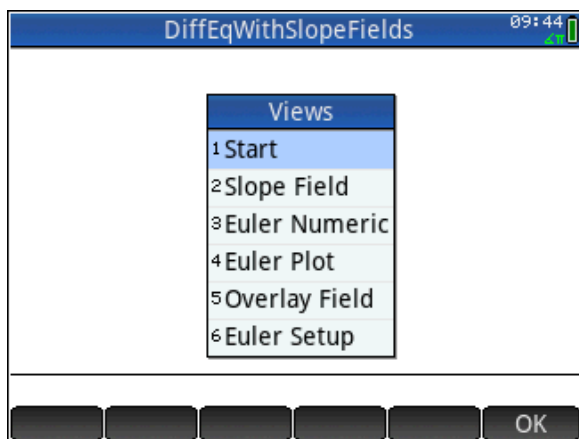
Differential equations, slope fields, Euler's Method

## Activity:

Students connect to the HP Classroom Network, and the teacher distributes the DiffEqWithSlopeFields App. Once you have the App, press , and select DiffEqWithSlopeFields. The App is based on the built-in Statistics 2Var App. Press  and select **1 Start** to see a note describing how to use the App. Many of the user interactions occur from the **View** menu. Notice that the App uses a slope expression in terms of X and /or Y that is defined in the Function App as F1(X).



So, press , and select Function. Define  $F1(X) = \sqrt{X}$ . By doing so, within the DiffEqWithSlopeFields App, you will be working with the differential equation  $\frac{dy}{dx} = \sqrt{x}$ . Press , and select DiffEqWithSlopeFields. Then press . The DiffEqWithSlopeFields App gives you the opportunity to explore differential equations graphically, numerically, and symbolically.



First, let's look at the slope field. Before you draw the slope field, think about the differential equation.

1. Will the slope segments in the slope field be positive or negative?

\_\_\_\_\_

2. What should the slope field show for  $X < 0$ ? \_\_\_\_\_

Select **2 Slope Field** and look at the slope field.

3. What happens to the slopes as X increases? \_\_\_\_\_

Notice that the slopes are the same within the same column and change as you move across the field (that is, slope depends only on  $X$ ). Later, you will see an example where the slopes are the same within the same row and change as you move up and down in the field (slope depends only on  $Y$ ). In other examples, slopes change as you move vertically or horizontally in the field (slope depends on both  $X$  and  $Y$ ).

Now let's take a quick look at how Euler's Method works. Though this is not a topic covered in AP Calculus AB, doing so let's us explore a numeric method for solving differential equations. Moreover, Euler's Method as you will see is really just a running left-hand Riemann Sum (a topic which is a part of AP Calculus AB).

The idea of Euler's Method is this. At any point in the plane (and we start at some chosen point, the initial condition of the differential equation), we can calculate slope using the differential equation. With that point and slope, we find the equation of a line, and use that equation to find another point a distance of  $\Delta X$  away (we keep  $\Delta X$  constant). At the new point, we use the differential equation to calculate slope again, and then find an equation of another line. We use that line to find a third point a distance of  $\Delta X$  away. We repeat the process until we reach some predetermined final value of  $X$ .

Using a linear equation to calculate a new value for  $Y$  each time comes down to this formula:

$$Y_{new} = Y_{old} + slope \cdot \Delta X.$$

Let's apply this process with the differential equation  $\frac{dy}{dx} = \sqrt{x}$  with an initial condition of  $(1, 0)$ .

We'll approximate  $y$  when  $x = 9$ , using two equal steps. This makes  $\Delta x = 4$ . We start at  $(1, 0)$ .


4. What is the slope at  $(1, 0)$ ? \_\_\_\_\_

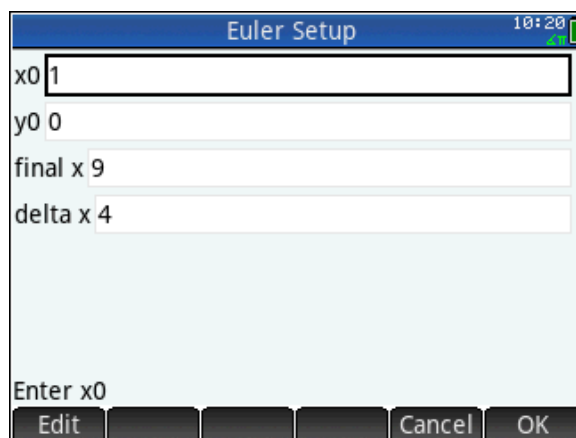
Now using the formula  $Y_{new} = Y_{old} + slope \cdot \Delta X$  to calculate the new  $y$ , we have  $Y_{new} = 0 + 1 \cdot 4 = 4$ . Our new point is  $(5, 4)$ .

5. What is the slope at  $(5, 4)$ ? \_\_\_\_\_

Again using the formula  $Y_{new} = Y_{old} + slope \cdot \Delta X$ , we calculate the new  $y$ . Use the **Home** screen to calculate the new  $Y$ .

6. What is the approximate value of  $Y$  when  $X = 9$ ? \_\_\_\_\_

Press  and select **6 Euler Setup**. Enter the parameters shown in the screen at right. These were determined by the initial condition,  $(1, 0)$ , the final  $X$ , 9, and the number of steps, 2. Then select **3 Euler Numeric**. Compare the value displayed with your answer to question 6. They should be the





same. Select **4 Euler Plot** to see a scatter plot of the three points. Trace on the scatter plot, and you'll see the coordinates of the three points.

Apply Euler's Method with the same differential equation and initial condition, but this time using 4 steps instead of two.

7. Complete this table. The first step has been done for you.

| $(X_{old}, Y_{old})$ | Slope | $Y_{new} = Y_{old} + slope \cdot \Delta X$ | $(X_{new}, Y_{new})$ |
|----------------------|-------|--|----------------------|
| (1,0)                | 1     | $Y_{new} = 0 + 1 \cdot 2$                  | (3,2)                |
| (3,2)                |       |  |                      |
|                      |       |  |                      |
|                      |       |  |                      |



Press  and select **6 Euler Setup**. Change  $\Delta x$  to 2. Then select **3 Euler Numeric**. Compare the value displayed with your final  $y$ . They should be the same. Select **4 Euler Plot** to see a scatter plot of the points. Adjust the viewing window appropriately, then press  and select **5 Overlay Field**. You'll see a slope field on top of the Euler Plot. Trace on the scatter plot to check your answers.

Now, go back to the Riemann Sum activity and compare these calculations with those that answered questions 3 and 7.

8. What do you notice? \_\_\_\_\_

Solve this differential equation analytically using separation of variables.

9. What is the solution? \_\_\_\_\_

Press  and enter the solution into **Fit1**. Press  and select **5 OverlayField**. Behold! A graphic, numeric, and analytic perspective on the solution of a differential equation!

Press  to trace on the Fit. The press 9 jump to the point where  $x = 9$ .

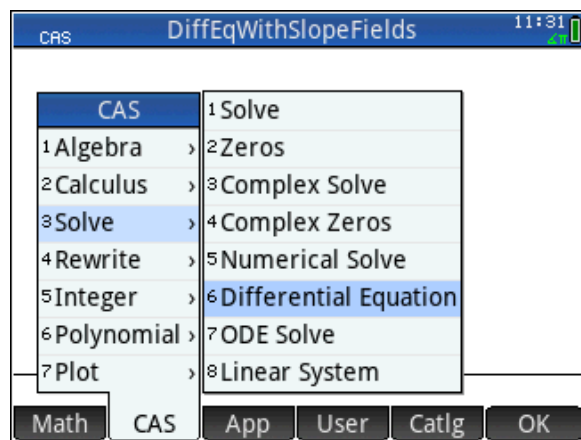
10. What is the exact value of  $y$  when  $x = 9$ ? \_\_\_\_\_

Notice that our slope function is increasing on the interval from  $x = 1$  to  $x = 9$ . This means that the solution curve is concave up.

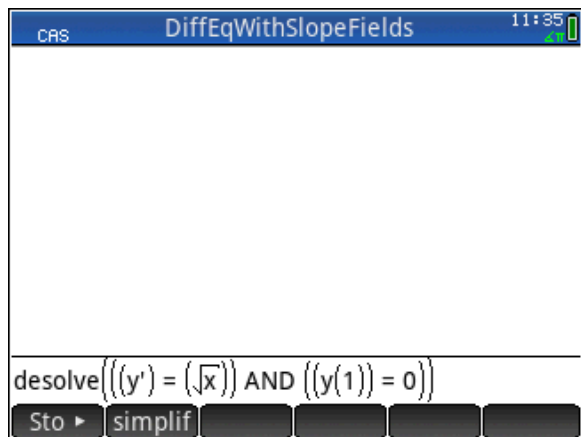
11. Whenever the solution curve is concave up, will Euler's Method result in an over- or under-approximation of the exact function value? \_\_\_\_\_

Recall from the Riemann Sum activity that when the integrand was increasing, left hand sums will under-approximate the exact area under the curve. Notice the connection!

HP Prime will solve differential equations analytically. Press **CAS**. Then press **Math** **CAS** and select **2 Solve** and **5 Differential Equation**.



Enter the command shown on the figure at right. Use the **ALPHA** shift key to enter y, and **Shift** **Vars** (Chars) to find the ' character. The word "and" can be entered in lower case.



## Extension

Use the *DiffEqWithSlopeFields* app to explore and solve this differential equation:

12.  $\frac{dy}{dx} = \frac{1-y}{2}$  with initial condition  $(-3, -4)$ .

(a) Approximate  $y$  when  $x = 3$  using Euler's method with 12 equal steps. \_\_\_\_\_

(b) Solve the differential equation analytically, and overlay the solution, slope field, and Euler Plot.

\_\_\_\_\_

(c) Does Euler's Method under- or over-approximate the solution? Why?

\_\_\_\_\_

\_\_\_\_\_

\_\_\_\_\_



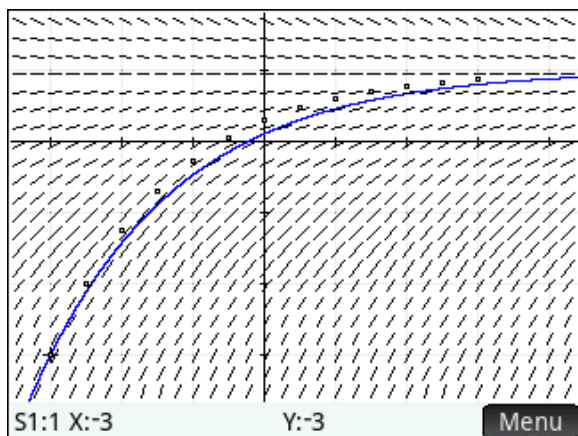
## Answers

1. Positive
2. Since  $\frac{dy}{dx}$  is undefined when  $x < 0$ , there should be no slope segments there.
3. As  $x$  increases, the slopes increase.
4. At  $(1,0)$  the slope is 1
5. At  $(5,4)$  the slope is  $\sqrt{5} \approx 2.2361$
6. When  $X = 9$ ,  $Y$  is approximately 12.9443
- 7.

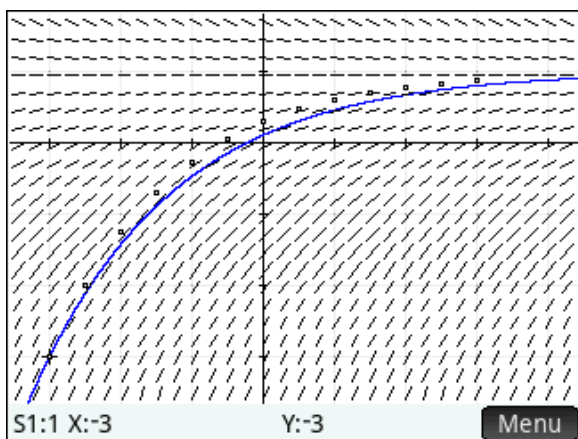
| $(X_{old}, Y_{old})$ | Slope      | $Y_{new} = Y_{old} + slope \cdot \Delta X$ | $(X_{new}, Y_{new})$ |
|----------------------|------------|--|----------------------|
| $(1,0)$              | 1          | $Y_{new} = 0 + 1 \cdot 2$                  | $(3,2)$              |
| $(3,2)$              | $\sqrt{3}$ | $Y_{new} = 2 + \sqrt{3} \cdot 2$           | $(5,5.4641)$         |
| $(5,5.4641)$         | $\sqrt{5}$ | $Y_{new} = 5.4641 + \sqrt{5} \cdot 2$      | $(7,9.9362)$         |
| $(7,9.9362)$         | $\sqrt{7}$ | $Y_{new} = 9.9362 + \sqrt{7} \cdot 2$      | $(9,15.2277)$        |

8. The calculation of the left hand Riemann Sum to approximate  $\int_1^9 \sqrt{x} dx$  using 4 rectangles of equal width is identical to the Euler's Method calculations in the table.
9. The solution is  $y = \frac{2}{3} x^{3/2} - \frac{2}{3}$ .
10. When  $x = 9$ ,  $y = 17.3333$  (notice that this is the same as  $\int_1^9 \sqrt{x} dx$ ).
11. When the solution curve is concave up, Euler's Method will result in an under-approximation of the exact function value.
12. (a) The approximation is 0.8733

(b) The solution is  $y = 1 - 4e^{-\frac{x}{2}}$ . Here's the graph



(c) Euler's Method over-approximates the solution, since the graph of the solution is concave down.



## Primed for the AP Exams

### HP Prime Functionality Introduced:

Evaluating derivatives and integrals numerically; storing intersection values in global variables

### AP Calculus Content:

Area of a region; local extrema; points of inflection

### Activity:

For the AP Calculus examinations, students are expected to have a calculator with this functionality:

- Graph a function in an arbitrary window
- Solve an equation numerically
- Evaluate a derivative numerically
- Evaluate an integral numerically

This activity illustrates how to do these things with HP Prime in an efficient way.

The first expectation, graphing a function in an arbitrary window, is one where Prime truly shines. When a window is given in a problem, students should use **Shift** **Plot** **Setup** (Plot Setup) to enter values for the viewing window. For other problems, it is usually a good idea to start in the default window, **View** **Copy** **4 Decimal** and use the scrolling or pinching gestures with the touch screen to adjust as needed.

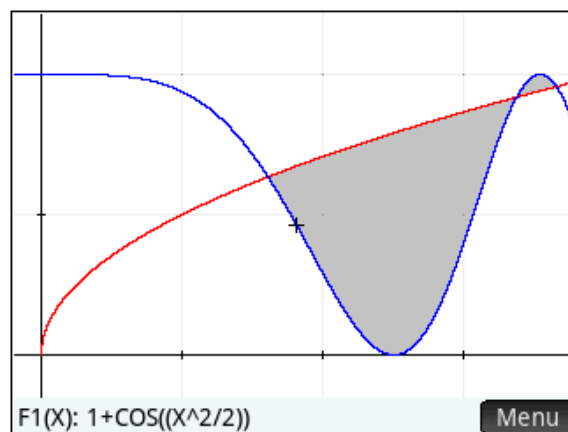
There are several ways to solve an equation with HP Prime. Students could use the **Solve** App, the solve() command in the CAS, or the Root or Intersection features in the Function Plot environment. Here, we will illustrate the latter process in the context of finding the area of a region.

#### Problem 1

The shaded region  $R$  shown in the graph at right is bounded by the graphs of

$$f(x) = 1 + \cos\left(\frac{x^2}{2}\right) \text{ and } g(x) = \sqrt{x}.$$

the area of  $R$ .



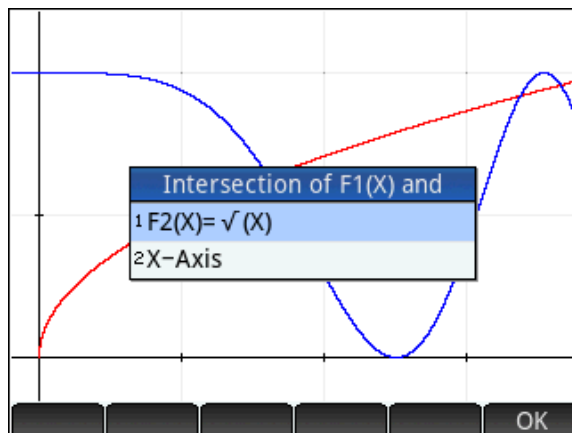
## Prime Solution

Start the Function App, if necessary, and press **Shift** **Plot** **Plot Setup** (Plot Setup). Enter a viewing window consistent with the given graph. A horizontal view of  $[-0.2, 4]$  and a vertical view of  $[-0.2, 2.3]$  would be appropriate. Press **Symb** **Plot Setup** and enter

$$f(x) = 1 + \cos\left(\frac{x^2}{2}\right) \text{ as } F1(X) \text{ and } g(x) = \sqrt{x} \text{ as } F2(X),$$

then press **Plot** **Plot Setup**. In order to find the area, we'll need to find the x-coordinates of the three points where the graphs intersect. We'll store these into global variables so we can use them later. First, tap the screen near the leftmost point of intersection to move the trace there. Then press **Menu** and then press **Fcn** and select **2 Intersection**. When prompted, select the intersection of F1(X) and F2(X). HP Prime finds the intersection closest to the cursor (here,  $X = 1.61126$ ). Immediately after the intersection is found, press **Settings** and press **X** **Sto** **A**. Press **Plot** **Plot Setup**, tap the screen near the second point of intersection, and repeat the process, storing the resulting X coordinate in B. Store the third intersection in C.

Now we'll evaluate integrals to get the area. We could do this either on the Plot screen, or in Home. We'll show how to do it in Home, since students need to write down their setups on the AP Exam anyway. Press **Units** **Units** and tap the template for the definite integral, in the second row, fourth column. Use the cursor keys to navigate in the template, and enter the integral as shown. Then press **Ans** **+**, select the integral template again, and enter the integral that gives the area of the second part of the region. See the screen at right. If your answer is smaller than 1.661, you probably subtracted in the integrand in reverse order.



| Function |               | 07:56 |
|----------|---------------|-------|
| X→A      | 1.61125731881 |       |
| X→B      | 3.37815035811 |       |
| X→C      | 3.66123825079 |       |
| Sto      |               |       |

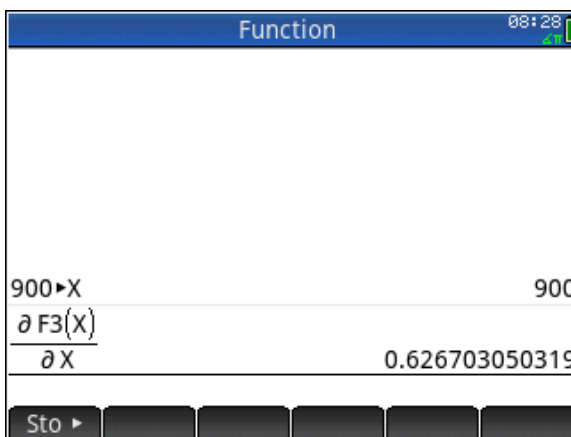
| Function                          |               | 08:07 |
|-----------------------------------|---------------|-------|
| X→A                               | 1.61125731881 |       |
| X→B                               | 3.37815035811 |       |
| X→C                               | 3.66123825079 |       |
| $\int_A^B F2(X) - F1(X) dX$       | 1.66117541473 |       |
| Ans + $\int_B^C F1(X) - F2(X) dX$ | 1.68372359816 |       |
| Sto                               |               |       |

## Problem 2

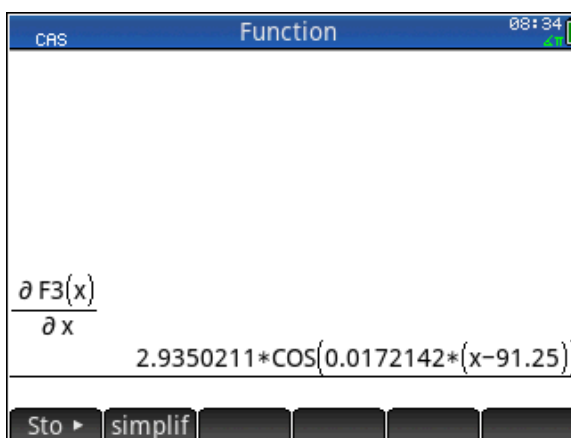
Finally, here's a quick problem that illustrates how to evaluate a derivative numerically with HP Prime. Recall the function from the Sunrise-Sunset (Precalculus) activity that modeled the number of minutes of daylight as a function of the number days since the winter solstice. We had  $L(x) = 170.5 \cdot \sin(0.0172142 \cdot (x - 91.25)) + 730.5$ . On the 900<sup>th</sup> day after the winter solstice, was the amount of daylight increasing or decreasing? Justify your answer.

### Prime Solution

Enter the formula for  $L(x)$  in the function App as F3(X). Then on the Home screen, store 900 in the variable X by pressing 900 **Sto** **X**. Press **Units** and tap the template for the derivative, in the first row, fourth column. In the top space, enter F3(X), and in the bottom, enter X. Press **Enter** to evaluate. We see that  $L'(900) = 0.627 > 0$ . Since this is positive, the amount of daylight is increasing on day 900.



HP Prime has the ability to calculate the derivative symbolically as well. The place to do this is in the CAS. We can use the Copy and Paste feature to copy results back to the Function App. Press **CAS Settings**, and enter the expression shown in the screen at right. Note again that the CAS uses lower case variables.



Incidentally, remember the Sunrise-Sunset (Calculus) activity? The value we got for the amplitude of the fit to the scatter plot of the average rates of change of  $L(X)$  versus  $X$  was  $A = 2.625$ . Compare this to what the chain rule gives us when we differentiate our model,  $L(X)$ , 2.935. Pretty close!