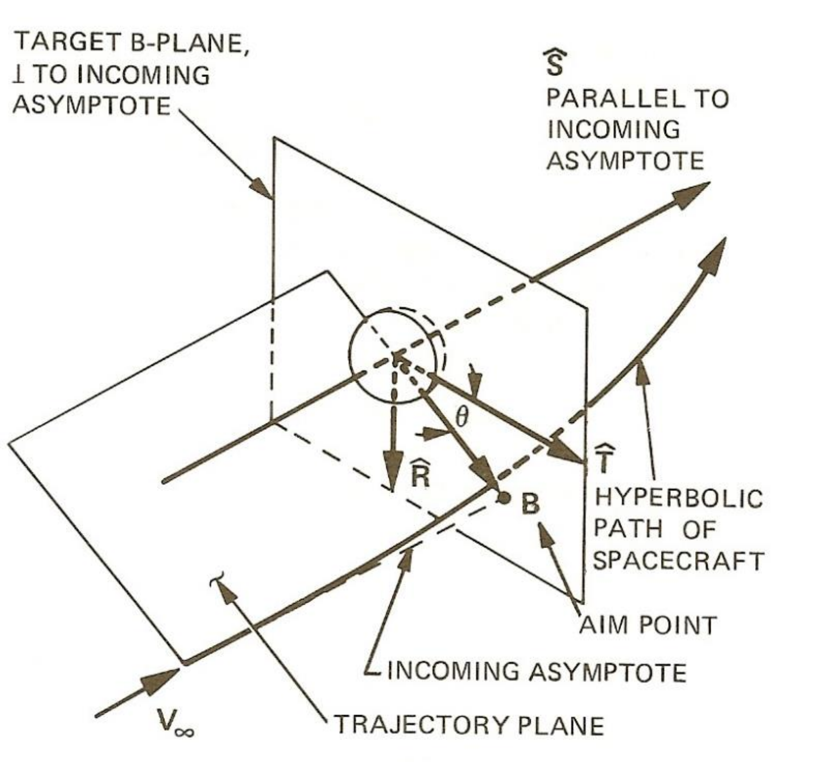
Appendix D

B-Plane Geometry, Coordinates and Targeting

The derivation of B-plane coordinates is described in the classic JPL reports, “A Method of Describing Miss Distances for Lunar and Interplanetary Trajectories” and “Some Orbital Elements Useful in Space Trajectory Calculations”, both by William Kizner. The following diagram illustrates the geometry of the B-plane coordinate system.



The arrival asymptote unit vector $\hat{\mathbf{S}}$ is given by

$$\hat{\mathbf{S}} = \begin{Bmatrix} \cos\delta_\infty \cos\alpha_\infty \\ \cos\delta_\infty \sin\alpha_\infty \\ \sin\delta_\infty \end{Bmatrix}$$

where δ_∞ and α_∞ are the declination and right ascension of the asymptote of the incoming hyperbola at the arrival planet.

B-plane calculations

This section describes the conversion of inertial coordinates of an arrival or departure hyperbola to fundamental B-plane coordinates and vectors.

angular momentum vector

$$\mathbf{h} = \mathbf{r} \times \mathbf{v}$$

radius rate

$$\dot{r} = \mathbf{r} \bullet \mathbf{v} / |\mathbf{r}|$$

semi-parameter

$$p = \frac{h^2}{\mu}$$

semimajor axis

$$a = \frac{r}{\left(2 - \dfrac{r\,v^2}{\mu}\right)}$$

orbital eccentricity

$$e = \sqrt{1 - p/a}$$

true anomaly

$$\cos v = \frac{p - r}{e\,r} \qquad \sin v = \frac{\dot{r}\,h}{e\mu}$$

B-plane magnitude

$$B = r_p \sqrt{1 + \frac{2\mu}{r_p V_\infty^2}} = \frac{\mu}{V_\infty^2} \sqrt{\left(1 + V_\infty^2 \frac{r_p}{\mu}\right)^2 - 1}$$

fundamental vectors

$$\hat{\mathbf{z}} = \frac{r\,\mathbf{v} - \dot{r}\,\mathbf{r}}{h}$$

$$\hat{\mathbf{p}} = \cos\theta\,\hat{\mathbf{r}} - \sin\theta\,\hat{\mathbf{z}} \qquad \hat{\mathbf{q}} = \sin\theta\,\hat{\mathbf{r}} + \cos\theta\,\hat{\mathbf{z}}$$

S vector

$$\mathbf{S} = -\frac{a}{\sqrt{a^2 + b^2}}\hat{\mathbf{p}} + \frac{b}{\sqrt{a^2 + b^2}}\hat{\mathbf{q}} \qquad \text{where } b = \sqrt{p\left|a\right|}$$

B vector

$$\mathbf{B} = \frac{b^2}{\sqrt{a^2 + b^2}}\hat{\mathbf{p}} + \frac{ab}{\sqrt{a^2 + b^2}}\hat{\mathbf{q}}$$

T vector

$$\mathbf{T} = \frac{\left(S_y^2, -S_x^2, 0\right)^T}{\sqrt{S_x^2 + S_y^2}}$$

R vector

$$\mathbf{R} = \mathbf{S} \times \mathbf{T} = \left(-S_z T_y,\; S_z T_x,\; S_x T_y - S_y T_x\right)^T$$

where **r** and **v** are the spacecraft inertial position and velocity vectors and μ is the planet's gravitational constant.

Techniques for B-Plane targeting

This section describes several techniques for using B-plane coordinates to *target* to specific planetary or moon encounter conditions. These targets are usually formulated as equality mission constraints which are enforced while solving the trajectory optimization problem.

user-defined B-plane coordinates

For this targeting option, the two nonlinear equality constraints enforced by the nonlinear programming (NLP) algorithm are

$$(\mathbf{B}\bullet\mathbf{T})_p - (\mathbf{B}\bullet\mathbf{T})_u = 0 \qquad (\mathbf{B}\bullet\mathbf{R})_p - (\mathbf{B}\bullet\mathbf{R})_u = 0$$

where the *p* subscript refers to coordinates predicted by the software and the *u* subscript denotes coordinates provided by the user. The *predicted* B-plane coordinates are based on the planet-centered flight conditions at closest approach.

targeting to user-defined entry interface (EI) conditions with user-defined inclination

For this targeting option, the following equations can be used to determine the required B-plane components based on the user-defined EI targets. These targets are the inertial flight path angle and altitude relative to a spherical planet model.

$$\mathbf{B}\bullet\mathbf{T} = b_t \cos\theta \qquad \mathbf{B}\bullet\mathbf{R} = b_t \sin\theta$$

where

$$b_t = \cos\gamma_{ei}\sqrt{\frac{2\mu r_{ei}}{v_\infty^2} + r_{ei}^2} \quad \text{and} \quad \cos\theta = \frac{\cos i}{\cos\delta_\infty}$$

Note that this targeting option could be modified to use a user-defined B-plane angle instead of orbital inclination at the entry interface.

In these equations, γ_{ei} is the user-defined flight path angle at the entry interface, r_{ei} is the arrival planet-centered radius at the entry interface (sum of planet equatorial radius plus user-defined EI altitude) and *i* is the user-defined orbital inclination.

Also, these dot product mission constraints can be expressed in terms of the *x*, *y* and *z* components of the spacecraft's inertial position and velocity vectors according to

$$\mathbf{B}\bullet\mathbf{T} = \frac{r_x v_y - r_y v_x}{\sqrt{v_x^2 + v_y^2}} \qquad \mathbf{B}\bullet\mathbf{R} = \frac{\left(r_x v_x + r_y v_y\right)v_z - \left(v_x^2 + v_y^2\right)r_z}{\sqrt{v_x^2 + v_y^2}\sqrt{v_x^2 + v_y^2 + v_z^2}}$$

targeting to a planet-centered grazing flyby with user-defined B-plane angle

The general expression for the periapsis radius of an encounter hyperbola at the arrival planet is given by

$$\tilde{r}_p = \frac{1}{\tilde{v}_\infty^2}\left(\sqrt{1 + \tilde{b}_\infty^2 \tilde{v}_\infty^4} - 1\right)$$

where the *normalized* quantities are

$\tilde{r}_p =$ normalized periapsis radius $= r_p / r_m$

$\tilde{b}_\infty =$ normalized b-plane magnitude $= b_\infty / r_m$

$\tilde{v}_\infty =$ normalized v-infinity speed $= v_\infty / v_{lc}$

$v_{lc} =$ local circular speed at Mars $= \sqrt{\mu_m / r_m}$

$r_m =$ radius of Mars

$\mu_m =$ gravitational constant of Mars

For a grazing flyby, $\tilde{r}_p = 1$ and the normalized B-plane distance or <u>offset</u> is equal to

$$\tilde{b}_\infty = \sqrt{1 + \frac{2}{\tilde{v}_\infty^2}}$$

Therefore, the required B-plane equality constraints are computed from

$$\mathbf{B} \bullet \mathbf{T} = b_\infty \cos\theta \qquad \mathbf{B} \bullet \mathbf{R} = b_\infty \sin\theta$$

where θ is the user-defined B-plane angle of the arrival trajectory. Please note that the B-plane angle is measured positive clockwise from the **T** axis of the B-plane coordinate system. The two equality constraints for this option are simply the difference between the predicted and required $\mathbf{B} \bullet \mathbf{T}$ and $\mathbf{B} \bullet \mathbf{R}$ components.

targeting to a planet-centered node/apse alignment trajectory

This targeting option determines when a spacecraft is simultaneously at a nodal crossing and periapsis of the arrival planet encounter hyperbola. The B-plane angle required for node-apse alignment is given by

$$\theta = \sin^{-1}\left(\frac{\sin\phi_h \hat{S}_z}{\cos\phi_h \sqrt{\hat{S}_x^2 + \hat{S}_y^2}} \right)$$

where $\phi_h = \sin^{-1}(1/e)$. This relationship is developed in the thesis of R. H. See.

Also noted in the thesis of R. H. See, the minimum and maximum values of orbital inclination can be determined from the *x* and *y* components of the unit $\hat{\mathbf{S}}$ vector according to

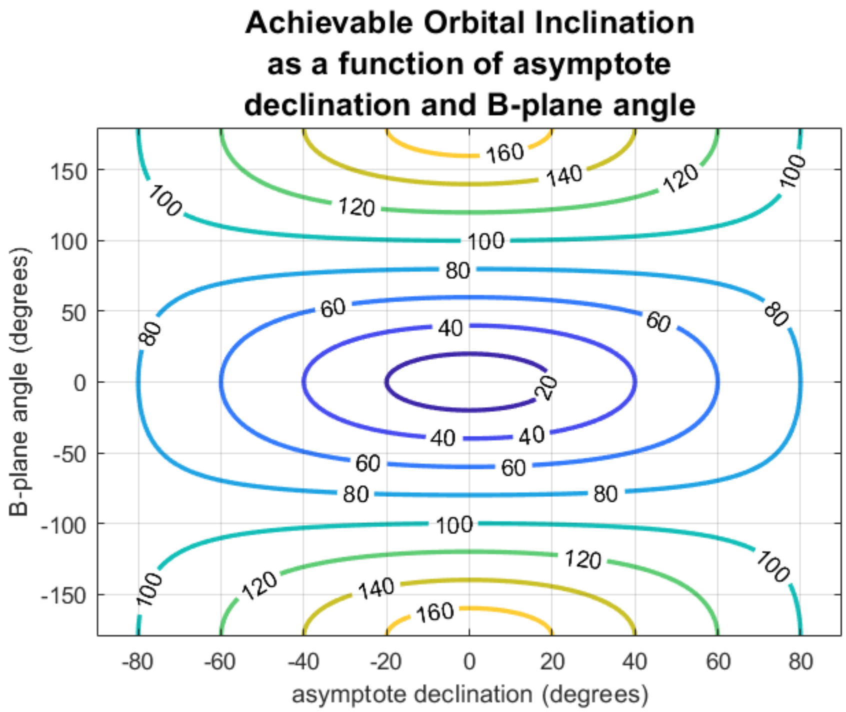
$$i_{\min} = \cos^{-1}\left(\sqrt{\hat{S}_x^2 + \hat{S}_y^2} \right) \qquad i_{\max} = \cos^{-1}\left(-\sqrt{\hat{S}_x^2 + \hat{S}_y^2} \right)$$

For an arrival hyperbola that lies in the equatorial plane, the following condition must be true

$$i_{\min} = 0 \rightarrow \sqrt{\hat{S}_x^2 + \hat{S}_y^2} = 1$$

Furthermore, if the *z* component of the incoming v-infinity vector is positive, $\mathbf{V}_{\infty_z} > 0$, an equatorial areocentric orbit is not possible.

The relationship between orbital inclination *i*, B-plane angle θ and asymptote declination δ of an incoming or outgoing hyperbola is given by $\cos i = \cos\theta\cos\delta$. The following is a contour plot illustrating the *achievable* inclination as a function of B-plane angle and declination.



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