Appendix A

Trajectory Modeling and Targeting in the Modified Equinoctial Orbital Elements System

The modified equinoctial orbital elements (MEE) are a set of orbital elements that are useful for trajectory analysis and optimization. They are valid for circular, elliptic, and hyperbolic orbits. This feature is important for trajectories that may transition to and from elliptic and hyperbolic orbits. These equations exhibit no singularity for zero eccentricity and orbital inclinations equal to 0 and 90 degrees. However, two components of the orbital element set are singular for an orbital inclination of 180^0.

The relationship between direct modified equinoctial and classical orbital elements is defined by the following definitions:

$$p = a\left(1-e^2\right) \qquad f = e\cos\left(\omega+\Omega\right) \qquad g = e\sin\left(\omega+\Omega\right)$$

$$h = \tan\left(i/2\right)\cos\Omega \qquad k = \tan\left(i/2\right)\sin\Omega \qquad L = \Omega+\omega+\theta$$

where

$p =$ semiparameter

$a =$ semimajor axis

$e =$ orbital eccentricity

$i =$ orbital inclination

$\omega =$ argument of periapsis

$\Omega =$ right ascension of the ascending node

$\theta =$ true anomaly

$L =$ true longitude

The relationship between classical and modified equinoctial orbital elements is summarized as follows:

semimajor axis

$$a = \frac{p}{1-f^2-g^2}$$

orbital eccentricity

$$e = \sqrt{f^2+g^2}$$

orbital inclination

$$i = 2\tan^{-1}\left(\sqrt{h^2+k^2}\right)$$

argument of periapsis

$$\omega = \tan^{-1}\left(g, f\right) - \tan^{-1}\left(k, h\right)$$

$$\sin\omega = \frac{g\,h - f\,k}{e\tan\left(i/2\right)} \qquad \cos\omega = \frac{f\,h + g\,k}{e\tan\left(i/2\right)}$$

right ascension of the ascending node (RAAN)

$$\Omega = \tan^{-1}(k, h)$$

$$\sin\Omega = \frac{k}{\tan(i/2)} \qquad \cos\Omega = \frac{h}{\tan(i/2)}$$

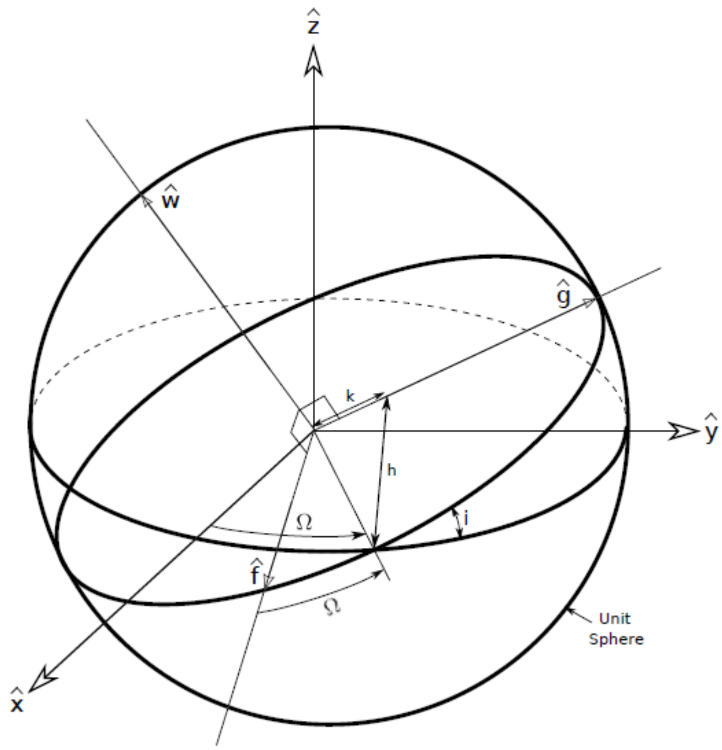
true anomaly

$$\theta = L - (\omega + \Omega) = L - \tan^{-1}(g, f)$$

$$\sin\theta = \frac{1}{e}(f \sin L - g \cos L) \qquad \cos\theta = \frac{1}{e}(f \cos L + g \sin L)$$

In these expressions, an inverse tangent expression of the form $\theta = \tan^{-1}(a, b)$ denotes a four quadrant evaluation where $a = \sin\theta$ and $b = \cos\theta$.

The following figure illustrates the orientation of the Earth-centered-inertial (ECI) and MEE coordinate systems on a unit sphere. The unit vectors in the ECI system are $\hat{x}, \hat{y}, \hat{z}$ and the unit vectors in the MEE system are $\hat{f}, \hat{g}, \hat{w}$.



$$\hat{\mathbf{f}} = \begin{Bmatrix} 1 - k^2 + h^2 \\ 2kh \\ 2k \end{Bmatrix} \qquad \hat{\mathbf{g}} = \begin{Bmatrix} 2kh \\ 1 + k^2 - h^2 \\ 2h \end{Bmatrix}$$

The mathematical relationships between an inertial state vector and the corresponding modified equinoctial elements are summarized as follows:

position vector

$$\mathbf{r} = \begin{bmatrix} \dfrac{r}{s^2}\left(\cos L + \alpha^2 \cos L + 2hk \sin L\right) \\ \dfrac{r}{s^2}\left(\sin L - \alpha^2 \sin L + 2hk \cos L\right) \\ \dfrac{2r}{s^2}\left(h \sin L - k \cos L\right) \end{bmatrix}$$

velocity vector

$$\mathbf{v} = \begin{bmatrix} -\dfrac{1}{s^2}\sqrt{\dfrac{\mu}{p}}\left(\sin L + \alpha^2 \sin L - 2hk \cos L + g - 2fhk + \alpha^2 g\right) \\ -\dfrac{1}{s^2}\sqrt{\dfrac{\mu}{p}}\left(-\cos L + \alpha^2 \cos L + 2hk \sin L - f + 2ghk + \alpha^2 f\right) \\ \dfrac{2}{s^2}\sqrt{\dfrac{\mu}{p}}\left(h \cos L + k \sin L + fh + gk\right) \end{bmatrix}$$

where

$$\alpha^2 = h^2 - k^2 \qquad s^2 = 1 + h^2 + k^2 \qquad r = \frac{p}{w} \qquad w = 1 + f \cos L + g \sin L$$

Equations of orbital motion

The system of nonlinear, first-order modified equinoctial equations of orbital motion with respect to time is given by the next six equations.

$$\dot{p} = \frac{dp}{dt} = \frac{2p}{w}\sqrt{\frac{p}{\mu}}\Delta_t$$

$$\dot{f} = \frac{df}{dt} = \sqrt{\frac{p}{\mu}}\left[\Delta_r \sin L + \left[\left(w+1\right)\cos L + f\right]\frac{\Delta_t}{w} - \left(h \sin L - k \cos L\right)\frac{g\Delta_n}{w}\right]$$

$$\dot{g} = \frac{dg}{dt} = \sqrt{\frac{p}{\mu}}\left[-\Delta_r \cos L + \left[\left(w+1\right)\sin L + g\right]\frac{\Delta_t}{w} + \left(h \sin L - k \cos L\right)\frac{f\Delta_n}{w}\right]$$

$$\dot{h} = \frac{dh}{dt} = \sqrt{\frac{p}{\mu}}\frac{s^2\Delta_n}{2w}\cos L \qquad \dot{k} = \frac{dk}{dt} = \sqrt{\frac{p}{\mu}}\frac{s^2\Delta_n}{2w}\sin L$$

$$\dot{L} = \frac{dL}{dt} = \sqrt{\mu p}\left(\frac{w}{p}\right)^2 + \frac{1}{w}\sqrt{\frac{p}{\mu}}\left(h \sin L - k \cos L\right)\Delta_n$$

where $\Delta_r, \Delta_t, \Delta_n$ are *non-two-body* perturbations in the radial, tangential and normal directions, respectively. For an Earth orbiting spacecraft, the radial direction is along the geocentric radius vector of the spacecraft measured positive in a direction away from the gravitational center, the tangential direction is perpendicular to this radius vector measured positive in the direction of orbital motion, and the normal direction is positive along the angular momentum vector of the spacecraft's orbit.

The equations of orbital motion can also be expressed in vector form as

$$\dot{\mathbf{y}} = \frac{d\mathbf{y}}{dt} = \mathbf{A}(\mathbf{y})\mathbf{P} + \mathbf{b}$$

where

$$\mathbf{A} = \left\{ \begin{array}{ccc} 0 & \dfrac{2p}{w}\sqrt{\dfrac{p}{\mu}} & 0 \\ \sqrt{\dfrac{p}{\mu}}\sin L & \sqrt{\dfrac{p}{\mu}}\dfrac{1}{w}\left[(w+1)\cos L + f\right] & -\sqrt{\dfrac{p}{\mu}}\dfrac{g}{w}\left[h\sin L - k\cos L\right] \\ -\sqrt{\dfrac{p}{\mu}}\cos L & \sqrt{\dfrac{p}{\mu}}\dfrac{1}{w}\left[(w+1)\sin L + g\right] & \sqrt{\dfrac{p}{\mu}}\dfrac{f}{w}\left[h\sin L - k\cos L\right] \\ 0 & 0 & \sqrt{\dfrac{p}{\mu}}\dfrac{s^2\cos L}{2w} \\ 0 & 0 & \sqrt{\dfrac{p}{\mu}}\dfrac{s^2\sin L}{2w} \\ 0 & 0 & \sqrt{\dfrac{p}{\mu}}\dfrac{1}{w}\left[h\sin L - k\cos L\right] \end{array} \right\}$$

$$\mathbf{b} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & \sqrt{\mu p}\left(\dfrac{w}{p}\right)^2 \end{bmatrix}^T.$$

The <u>total</u> non-two-body acceleration vector is given by

$$\mathbf{P} = \Delta_r \hat{\mathbf{i}}_r + \Delta_t \hat{\mathbf{i}}_t + \Delta_n \hat{\mathbf{i}}_n$$

where $\hat{\mathbf{i}}_r, \hat{\mathbf{i}}_t$ and $\hat{\mathbf{i}}_n$ are unit vectors in the radial, tangential and normal directions. These unit vectors can be computed from the inertial position vector $\mathbf{r}$ and velocity vector $\mathbf{v}$ according to

$$\hat{\mathbf{i}}_r = \frac{\mathbf{r}}{|\mathbf{r}|} \qquad \hat{\mathbf{i}}_n = \frac{\mathbf{r}\times\mathbf{v}}{|\mathbf{r}\times\mathbf{v}|} \qquad \hat{\mathbf{i}}_t = \hat{\mathbf{i}}_n \times \hat{\mathbf{i}}_r = \frac{(\mathbf{r}\times\mathbf{v})\times\mathbf{r}}{|\mathbf{r}\times\mathbf{v}||\mathbf{r}|}$$

For *unperturbed* two-body motion, $\mathbf{P} = 0$ and the first five equations of motion are simply $\dot{p} = \dot{f} = \dot{g} = \dot{h} = \dot{k} = 0$. Therefore, for two-body motion these modified equinoctial orbital elements are constant. The true longitude is often called the *fast variable* of this orbital element set.

The dynamical equations of motion with true longitude L as the independent variable can be determined from the following two vector expressions:

$$\dot{\mathbf{z}} = \frac{d\mathbf{z}}{dL}\frac{dL}{dt} = \mathbf{f}(\mathbf{z})$$

$$\frac{d\mathbf{z}}{dL} = \mathbf{f}(\mathbf{z})\left[\frac{dL}{dt}\right]^{-1} = \frac{1}{\dot{L}}\mathbf{f}(\mathbf{z})$$

where $\mathbf{z}$ is the vector of modified equinoctial orbital elements.

Propulsive thrust

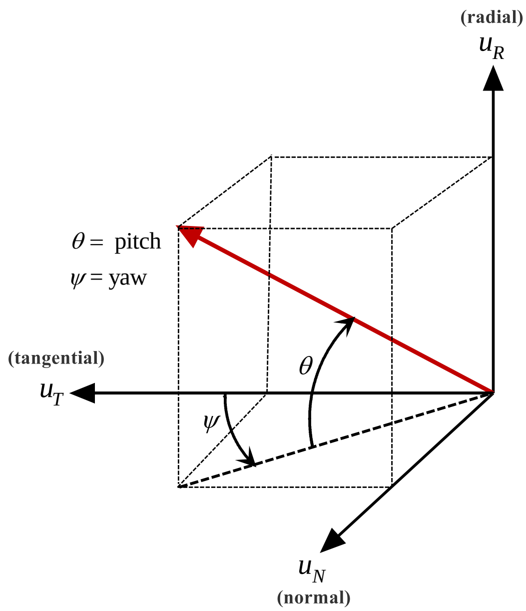
The acceleration due to propulsive thrust can be expressed as

$$\mathbf{a}_T = \eta\frac{T}{m}\hat{\mathbf{u}}$$

where T is the thrust magnitude, m is the spacecraft mass, $\hat{\mathbf{u}} = \begin{bmatrix} u_r & u_t & u_n \end{bmatrix}$ is the unit pointing thrust vector expressed in the spacecraft-centered radial-tangential-normal coordinate system, and η is the throttle setting. The components of the unit thrust vector can also be defined in terms of the in-plane pitch angle θ and the out-of-plane yaw angle ψ as follows:

$$u_r = \sin\theta \qquad u_t = \cos\theta\cos\psi \qquad u_n = \cos\theta\sin\psi$$

The pitch angle is positive above the "local horizontal" and the yaw angle is positive in the direction of the angular momentum vector.



The relationship between a unit thrust vector in the ECI coordinate system $\hat{\mathbf{u}}_{T_{ECI}}$ and the corresponding unit thrust vector in the modified equinoctial system $\hat{\mathbf{u}}_{T_{MEE}}$ is given by

$$\hat{\mathbf{u}}_{T_{ECI}} = \begin{bmatrix} \hat{\mathbf{i}}_r & \hat{\mathbf{i}}_t & \hat{\mathbf{i}}_n \end{bmatrix}\hat{\mathbf{u}}_{T_{MEE}}$$

This relationship can also be expressed as

$$\hat{\mathbf{u}}_{T_{ECI}} = [\mathbf{Q}]\hat{\mathbf{u}}_{T_{MEE}} = \begin{bmatrix} \hat{\mathbf{r}}_x & (\hat{\mathbf{h}}\times\hat{\mathbf{r}})_x & \hat{\mathbf{h}}_x \\ \hat{\mathbf{r}}_y & (\hat{\mathbf{h}}\times\hat{\mathbf{r}})_y & \hat{\mathbf{h}}_y \\ \hat{\mathbf{r}}_z & (\hat{\mathbf{h}}\times\hat{\mathbf{r}})_z & \hat{\mathbf{h}}_z \end{bmatrix}\hat{\mathbf{u}}_{T_{MEE}}$$

Finally, the transformation of the unit thrust vector in the ECI system to the modified equinoctial coordinate system is given by $\hat{\mathbf{u}}_{T_{MEE}} = [\mathbf{Q}]^T\,\hat{\mathbf{u}}_{T_{ECI}}$.

For the case of *tangential steering*

$$\hat{\mathbf{u}}_{T_{ECI}} = \left[(\hat{\mathbf{h}}\times\hat{\mathbf{r}})_x \quad (\hat{\mathbf{h}}\times\hat{\mathbf{r}})_y \quad (\hat{\mathbf{h}}\times\hat{\mathbf{r}})_z\right]^T.$$

Non-spherical Earth gravity

The non-spherical gravitational acceleration vector can be expressed as

$$\mathbf{g} = g_N\,\hat{\mathbf{i}}_N - g_r\,\hat{\mathbf{i}}_r$$

where

$$\hat{\mathbf{i}}_N = \frac{\hat{\mathbf{e}}_N - (\hat{\mathbf{e}}_N^T\,\hat{\mathbf{i}}_r)\hat{\mathbf{i}}_r}{\left\|\hat{\mathbf{e}}_N - (\hat{\mathbf{e}}_N^T\,\hat{\mathbf{i}}_r)\hat{\mathbf{i}}_r\right\|}$$

and

$$\hat{\mathbf{e}}_N = [0 \quad 0 \quad 1]^T$$

In these equations the north direction component is indicated by subscript *N* and the radial direction component is subscript *r*.

The contributions due to a *zonal* gravity model of order *n* are as follows:

$$g_N = -\frac{\mu\cos\phi}{r^2}\sum_{k=2}^{n}\left(\frac{R_e}{r}\right)^k P_k' J_k$$

$$g_r = -\frac{\mu}{r^2}\sum_{k=2}^{n}(k+1)\left(\frac{R_e}{r}\right)^k P_k J_k$$

where

$\mu =$ gravitational constant

$r =$ geocentric distance of the spacecraft

$R_e =$ equatorial radius of the Earth

$\phi =$ geocentric latitude

$J_k =$ zonal gravity coefficient

$P_k =$ $\mathrm{k^{th}}$ order Legendre polynomial

For a *zonal only* Earth gravity model, the east component is identically zero.

Finally, the zonal gravity perturbation contribution is $\mathbf{a}_g = \mathbf{Q}^T \mathbf{g}$ where $\mathbf{Q} = \begin{bmatrix} \hat{\mathbf{i}}_r & \hat{\mathbf{i}}_t & \hat{\mathbf{i}}_n \end{bmatrix}$.

For J_2 effects only, the three components are as follows:

$$\Delta_{J_{2_r}} = -\frac{3\mu J_2 R_e^2}{2r^4}\left[1 - \frac{12\left(h\sin L - k\cos L\right)^2}{\left(1+h^2+k^2\right)^2}\right]$$

$$\Delta_{J_{2_t}} = -\frac{12\mu J_2 R_e^2}{r^4}\left[\frac{\left(h\sin L - k\cos L\right)\left(h\cos L + k\sin L\right)}{\left(1+h^2+k^2\right)^2}\right]$$

$$\Delta_{J_{2_n}} = -\frac{6\mu J_2 R_e^2}{r^4}\left[\frac{\left(1-h^2-k^2\right)\left(h\sin L - k\cos L\right)}{\left(1+h^2+k^2\right)^2}\right]$$

Planetary perturbations

The general vector equation for *point-mass* perturbations such as the Moon or planets is given by

$$\ddot{\mathbf{r}} = -\sum_{j=1}^{n} \mu_j \left[\frac{\mathbf{d}_j}{d_j^3} + \frac{\mathbf{s}_j}{s_j^3}\right]$$

In this equation, $\mathbf{s}_j$ is the vector from the primary body to the secondary body *j*, μ_j is the gravitational constant of the secondary body and $\mathbf{d}_j = \mathbf{r} - \mathbf{s}_j$, where **r** is the position vector of the spacecraft relative to the primary body.

To avoid numerical problems, use is made of Professor Richard Battin's $F\left(q\right)$ function given by

$$F\left(q_k\right) = q_k\left[\frac{3+3q_k+q_k^2}{1+\left(\sqrt{1+q_k}\right)^3}\right]$$

where

$$q_k = \frac{\mathbf{r}^T\left(\mathbf{r}-2\mathbf{s}_k\right)}{\mathbf{s}_k^T\mathbf{s}_k}$$

The acceleration due to other planets can now be expressed as

$$\ddot{\mathbf{r}} = -\sum_{k=1}^{n} \frac{\mu_k}{d_k^3}\left[\mathbf{r} + F\left(q_k\right)\mathbf{s}_k\right]$$

Finally, the perturbation due to secondary bodies in the modified equinoctial coordinate system is given by $\mathbf{a} = \left[\mathbf{Q}\right]^T \mathbf{t}$ where $\mathbf{Q} = \begin{bmatrix} \hat{\mathbf{i}}_r & \hat{\mathbf{i}}_t & \hat{\mathbf{i}}_n \end{bmatrix}$.

Targeting with modified equinoctial elements

This section describes techniques for "targeting" with modified equinoctial orbital elements by enforcing equality and inequality mission constraints during the trajectory optimization. Constraint formulations that enforce both the sine and cosine of a desired orbital element should be used whenever possible. This approach involves a combination of equality and inequality constraints and ensures that the "targeted" orbital element is in the correct quadrant.

To illustrate this technique, here are several examples for different values of argument of perigee and the corresponding mission constraints.

$$0^\circ < \omega < 90^\circ \rightarrow \begin{cases} \sin\omega > 0 \rightarrow g\,h - f\,k > 0 \\ f\,h + g\,k = e\tan(i/2)\cos\omega \end{cases}$$

$$\omega = 270^\circ \rightarrow \begin{cases} \sin\omega \le 0 \rightarrow g\,h - f\,k \le 0 \\ \cos\omega = 0 \rightarrow f\,h + g\,k = 0 \end{cases}$$

$$\omega = 178^\circ \rightarrow \begin{cases} g\,h - f\,k = e\tan(i/2)\sin\omega \\ \cos\omega \le 0 \rightarrow f\,h + g\,k \le 0 \end{cases}$$

The following is a *sign* table of the sine and cosine for each quadrant.

quadrant	sine	cosine
1	+	+
2	+	−
3	−	−
4	−	+

orbital eccentricity constraint

$$e = \sqrt{f^2 + g^2}$$

For a <u>circular</u> orbit, $f = g = 0$.

<u>*orbital inclination constraint*</u>

$$\tan\left(\frac{i}{2}\right) = \sqrt{h^2 + k^2}$$

For an <u>equatorial</u> orbit, $h = k = 0$.

<u>*argument of perigee constraints*</u>

$$g\,h - f\,k = e\sin\omega\tan(i/2) \rightarrow \sin\omega = \frac{g\,h - f\,k}{e\tan(i/2)}$$

$$f\,h + g\,k = e\cos\omega\tan(i/2) \rightarrow \cos\omega = \frac{f\,h + g\,k}{e\tan(i/2)}$$

right ascension of the ascending node constraints

$$k = \tan(i/2)\sin\Omega \rightarrow \sin\Omega = \frac{k}{\tan(i/2)}$$

$$h = \tan(i/2)\cos\Omega \rightarrow \cos\Omega = \frac{h}{\tan(i/2)}$$

true anomaly constraints

$$\theta = L - (\omega + \Omega) = L - \tan^{-1}(g, f)$$

In general,

$$\sin\theta = \frac{1}{e}(f \sin L - g \cos L) \qquad \cos\theta = \frac{1}{e}(f \cos L + g \sin L)$$

For a circular orbit,

$$\sin\theta = \sin L \cos\Omega - \cos L \sin\Omega \qquad \cos\theta = \cos L \cos\Omega + \sin L \sin\Omega$$

For a circular, equatorial orbit,

$$\theta = L \qquad \sin\theta = \sin L \qquad \cos\theta = \cos L$$

Targeting example

For a user-defined semimajor axis, orbital eccentricity and inclination, the set of modified equinoctial equality constraints are as follows:

$$p = \tilde{p} \qquad \sqrt{f^2 + g^2} = \tilde{e} \qquad \sqrt{h^2 + k^2} = \tan(\tilde{i}/2)$$

where the tilde indicates the value of the user-defined classical orbital element.

"On the Equinoctial Orbital Elements", R. A. Brouke and P. J. Cefola, *Celestial Mechanics*, Vol. 5, pp. 303-310, 1972.

"A Set of Modified Equinoctial Orbital Elements", M. J. H. Walker, B. Ireland and J. Owens, *Celestial Mechanics*, Vol. 36, pp. 409-419, 1985.

"Equinoctial Orbit Elements: Application to Optimal Transfer Problems", Jean A. Kechichian, AIAA 90-2976, AIAA/AAS Astrodynamics Conference, Portland, OR, 20-22 August 1990.