Frozen Orbit Design

This document describes two MATLAB scripts that can be used to design and analyze the behavior of frozen orbits. A frozen orbit is characterized by the absence of long-term changes in orbital eccentricity and argument of perigee. This type of orbit maintains almost constant altitude over any particular point on a planet's surface.

The design of frozen orbits involves selecting the correct value of orbital eccentricity and argument of perigee, for a given semimajor axis and orbital inclination, which satisfies the following system of nonlinear perturbation equations:

$$\frac{de}{dt} = \frac{3}{2}\frac{J_3 r_{eq}^3}{p^3}\left(1-e^2\right) n \sin i \cos\omega \left(\frac{5}{4}\sin^2 i - 1\right) = 0$$

$$\frac{d\omega}{dt} = \frac{3}{2}\frac{J_2 r_{eq}^2}{p^2} n \left(2 - \frac{5}{2}\sin^2 i\right) - \frac{3}{2}\frac{J_3 r_{eq}^3 \sin\omega}{p^3 e \sin i} n \left\{\left(\frac{5}{4}\sin^2 i - 1\right)\sin^2 i + e^2\left(1 - \frac{35}{4}\sin^2 i \cos^2 i\right)\right\} = 0$$

where

$a =$ semimajor axis

$e =$ orbital eccentricity

$i =$ orbital inclination

$\omega =$ argument of perigee

$p = a\left(1-e^2\right) =$ semiparameter

$r_{eq} =$ Earth equatorial radius

$\mu =$ Earth gravitational constant

$n = \sqrt{\mu / a^3} =$ mean motion

$J_2 =$ second gravity coefficient

$J_3 =$ third gravity coefficient

By solving this system, the perturbing effect of the J_2 even zonal gravity harmonic on eccentricity and argument of perigee are balanced by the J_3 odd zonal gravity harmonic. For argument of perigee values equal to 90 and 270 degrees, the eccentricity perturbation vanishes.

frozen1.m – required mean orbital eccentricity

This MATLAB application determines the mean orbital eccentricity required for a frozen orbit. The user inputs the mean semimajor axis and inclination and the program calculates the eccentricity using Brent's method to solve the single constraining nonlinear equation. The program also calculates and displays the real roots of the frozen eccentricity cubic equation:

$$a_1 e^3 + a_2 e^2 + a_3 e + a_4 = 0$$

The derivation of this equation can be found in the technical report, "Frozen Orbits in the J2 + J3 Problem", by Krystyna Kiedron and Richard Cook, AAS 91-426, AAS/AIAA Astrodynamics Specialist Conference, Durango, Colorado, August 19-22, 1991.

The coefficients of this equation are given by

$$a_1 = -\frac{3}{4} n \left(\frac{R}{a}\right)^2 J_2 \sin i \left(1 - 5\cos^2 i\right)$$

$$a_2 = \frac{3}{2} n \left(\frac{R}{a}\right)^3 J_3 \left(1 - \frac{35}{4} \sin^2 i \cos^2 i\right)$$

$$a_3 = -a_1$$

$$a_4 = \frac{3}{2} n \left(\frac{R}{a}\right)^3 J_3 \sin^2 i \left(\frac{5}{4} \sin^2 i - 1\right)$$

In these equations n is the mean motion, R is the equatorial radius of the Earth, a is the semimajor axis, and i is the orbital inclination.

The following is a typical user interaction application with this MATLAB script.

```
          program frozen1

< orbital eccentricity of frozen orbits >


mean orbital elements

please input the semimajor axis (kilometers)
(semimajor axis > 0)
? 8000

please input the orbital inclination (degrees)
(0 <= inclination <= 180)
? 45

     mean orbital elements

      sma (km)        eccentricity     inclination (deg)    argper (deg)
 8.0000000000e+003  6.5941377284e-004  4.5000000000e+001  9.0000000000e+001

     raan (deg)     true anomaly (deg)   arglat (deg)       period (min)
 0.0000000000e+000  0.0000000000e+000  9.0000000000e+001  1.1868468430e+002


 roots of the frozen eccentricity cubic equation

ans =

  -1.00241917246590
   0.99758348478212
   0.00065941377284
```

frozen2.m – long-term evolution of frozen orbits

This application allows the user to analyze the long-term behavior of frozen orbits by numerically integrating the equations of orbital motion subject to zonal gravity perturbations. The software can model up to 18 zonal coefficients and the user can plot the long-term behavior of eccentricity versus time, argument of perigee versus time, or argument of perigee versus orbital eccentricity.

The second-order, nonlinear differential equations of orbital motion subject to both the spherical and zonal components of the Earth's gravitation attraction are given by

$$\ddot{\mathbf{r}} = -\mu \frac{\mathbf{r}}{|\mathbf{r}|^3} + \sum_{n=2}^{N} J_n \mathbf{U}_n$$

where **r** is the inertial position vector of the satellite, J_n are the zonal coefficients of the Earth gravity field and the "perturbed" or non-spherical contributions are

$$U_{2_x} = \frac{3}{2}\mu \frac{r_{eq}^2}{r^5}\left(5\frac{z^2}{r^2} - 1\right)x$$

$$U_{2_y} = \frac{3}{2}\mu \frac{r_{eq}^2}{r^5}\left(5\frac{z^2}{r^2} - 1\right)y$$

$$U_{2_z} = \frac{3}{2}\mu \frac{r_{eq}^2}{r^5}\left(5\frac{z^2}{r^2} - 1\right)z$$

$$U_{3_x} = \frac{1}{2}\mu \frac{r_{eq}^3}{r^6}\left(35\frac{z^3}{r^3} - 15\frac{z}{r}\right)x$$

$$U_{3_y} = \frac{1}{2}\mu \frac{r_{eq}^3}{r^6}\left(35\frac{z^3}{r^3} - 15\frac{z}{r}\right)y$$

$$U_{3_z} = \frac{1}{2}\mu \frac{r_{eq}^3}{r^6}\left(35\frac{z^3}{r^3} - 30\frac{z}{r} + 3\frac{z}{r}\right)z$$

In these equations, μ is the gravitational constant of the Earth, r_{eq} is the equatorial radius of the Earth, r is the geocentric radius of the satellite and the inertial rectangular components of the position vector are x, y and z. The recurrence relationship for the *U's* is as follows:

$$U_{n_x} = \left(\frac{2n+1}{n}\right)\frac{r_{eq}}{r}\frac{z}{r}U_{n-1_x} - \left(\frac{n+1}{n}\right)\frac{r_{eq}^2}{r^2}U_{n-2_x}$$

$$U_{n_y} = \left(\frac{2n+1}{n}\right)\frac{r_{eq}}{r}\frac{z}{r}U_{n-1_y} - \left(\frac{n+1}{n}\right)\frac{r_{eq}^2}{r^2}U_{n-2_y}$$

$$U_{n_z} = \left(\frac{2n+1}{n}\right)\frac{r_{eq}}{r}\frac{z}{r}U_{n-1_z} - \left(\frac{n+1}{n}\right)\frac{r_{eq}^2}{r^2}U_{n-2_z}$$

The steps implemented in this algorithm are as follows:

(1) user inputs mean orbital elements

(2) convert mean orbital elements to osculating orbital elements and then to state vector

(3) propagate the second-order "zonal" orbital equations of motion

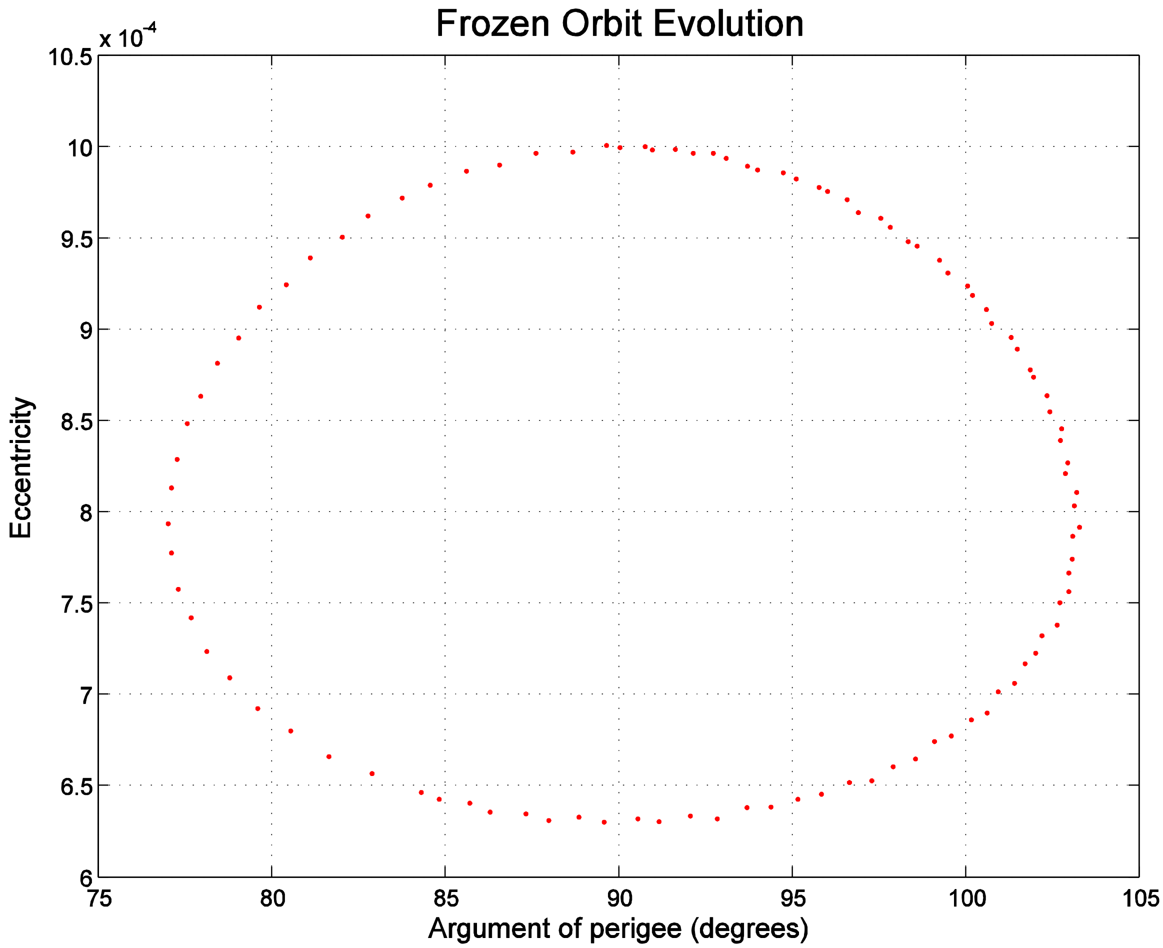
(4) convert the state vector at each step to mean orbital elements

(5) plot the user-requested orbital elements

The following is a typical user interaction with this MATLAB script.

```
          program frozen2

< orbital motion of frozen orbits >


please input the simulation period (days)
? 1000

please input the algorithm step size (minutes)
(a value between 1 and 2 is recommended)
? 2

please input the number of zonals to include
(0 <= zonals <= 18)
? 4

please input the graphics step size (days)
? 10

initial mean orbital elements

please input the semimajor axis (kilometers)
(semimajor axis > 0)
? 8000

please input the orbital eccentricity (non-dimensional)
(0 <= eccentricity < 1)
? .001

please input the orbital inclination (degrees)
(0 <= inclination <= 180)
? 60

please input the argument of perigee (degrees)
(0 <= argument of perigee <= 360)
? 90
```

The following is a graphics display of the argument of perigee versus orbital eccentricity for this example. The orbit evolution for this example is clockwise in this display starting from the top $\left(e = 0.001, \omega = 90^{\circ}\right)$. We can see that the frozen orbit eccentricity for this example is about `8e-4` for an argument of perigee of 90 degrees. The graphics step size for this example was 10 days.


Frozen Orbit Evolution
x 10^-4
10.5
10
9.5
9
8.5
8
7.5
7
6.5
6
Eccentricity
75
80
85
90
95
100
105
Argument of perigee (degrees)