The Hohmann Orbit Transfer

The coplanar circular orbit-to-circular orbit transfer was discovered by the German engineer Walter Hohmann in 1925 and described in his classic report, *The Attainability of Celestial Bodies*. The transfer consists of a velocity impulse on an initial circular orbit, in the direction of motion and collinear with the velocity vector, which propels the space vehicle into an elliptical transfer orbit. At a transfer angle of 180 degrees from the first impulse, a second velocity impulse or ΔV, also collinear and in the direction of motion, places the vehicle into a final circular orbit at the desired final altitude. The impulsive ΔV assumption means that the velocity, but not the position, of the vehicle is changed instantaneously. This is equivalent to a rocket engine with infinite thrust magnitude. The Hohmann formulation is the ideal and minimum energy solution to this type of time-free orbit transfer problem.

Coplanar Equations

For the coplanar Hohmann transfer both velocity impulses are confined to the orbital planes of the initial and final orbits. For a Hohmann transfer from a lower altitude orbit to a higher altitude circular orbit, the first impulse creates an elliptical transfer orbit with a perigee altitude equal to the altitude of the initial circular orbit and an apogee altitude equal to the altitude of the final orbit. The second impulse circularizes the transfer orbit at apogee. Both impulses are *posigrade* which means that they are in the direction of orbital motion.

We begin by defining three *normalized* radii as follows:

$$R_1 = \sqrt{2\frac{r_f}{r_i + r_f}} \qquad R_2 = \sqrt{\frac{r_i}{r_f}} \qquad R_3 = \sqrt{2\frac{r_i}{r_i + r_f}}$$

where r_i is the geocentric radius of the initial circular park orbit and r_f is the radius of the final circular mission orbit. The relationship between radius and initial orbit altitude h_i and the final orbit altitude h_f is as follows:

$$r_i = r_e + h_i$$

$$r_f = r_e + h_f$$

where r_e is the radius of the Earth.

The magnitude of the first impulse is

$$\Delta V_1 = V_{lc}\sqrt{1 + R_1^2 - 2R_1}$$

and is simply the difference between the speed on the initial orbit and the perigee speed of the transfer orbit. The scalar magnitude of the second impulse is

$$\Delta V_2 = V_{lc}\sqrt{R_2^2 + R_2^2 R_3^2 - 2R_2^2 R_3}$$

which is the difference between the speed on the final orbit and the apogee speed of the transfer ellipse.

In each of these ΔV equations V_{lc} is called the *local circular velocity*. It can be determined from

$$V_{lc} = \sqrt{\frac{\mu}{r_i}}$$

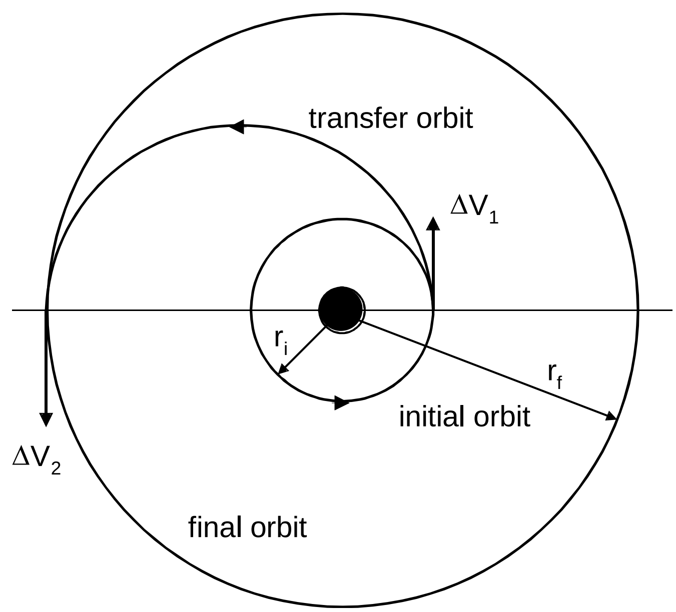
and represents the scalar speed in the initial orbit. In these equations μ is the gravitational constant of the central body. The transfer time from the first impulse to the second is equal to one half the orbital period of the transfer ellipse

$$\tau = \pi \sqrt{\frac{a^3}{\mu}}$$

where a is the semimajor axis of the transfer orbit and is equal to $\left(r_i + r_f\right)/2$. The orbital eccentricity of the transfer ellipse is

$$e = \frac{\max\left(r_i, r_f\right) - \min\left(r_i, r_f\right)}{r_f - r_i}$$

The following diagram illustrates the geometry of the coplanar Hohmann transfer.



Non-coplanar Equations

The non-coplanar Hohmann transfer involves orbital transfer between two circular orbits which have different orbital inclinations. For this problem the propulsive energy is minimized if we optimally partition the total orbital inclination change between the first and second impulses.

The scalar magnitude of the first impulse is

$$\Delta V_1 = V_{lc}\sqrt{1 + R_1^2 - 2R_1\cos\theta_1}$$

where θ_1 is the plane change associated with the first impulse. The magnitude of the second impulse is

$$\Delta V_2 = V_{lc}\sqrt{R_2^2 + R_2^2 R_3^2 - 2R_2^2 R_3\cos\theta_2}$$

where θ_2 is the plane change associated with the second impulse. These two equations are different forms of the law of cosines.

The total ΔV required for the maneuver is the sum of the two impulses as follows

$$\Delta V = \Delta V_1 + \Delta V_2$$

The relationship between the plane change angles is

$$\theta_t = \theta_1 + \theta_2$$

where θ_t is the total plane change angle between the initial and final orbits.

Optimizing the non-coplanar Hohmann transfer involves allocating the total plane change angle between the two maneuvers such that the total ΔV required for the mission is minimized. We can determine this answer by solving for the root of a derivative.

The partial derivative of the total ΔV with respect to the first plane change angle is given by:

$$\frac{\partial \Delta V}{\partial \theta_1} = \frac{R_1\sin\theta_1}{\sqrt{1 + R_1^2 - 2R_1\cos\theta_1}} - \frac{R_2^2 R_3\left(\sin\theta_t\cos\theta_1 - \cos\theta_t\sin\theta_1\right)}{\sqrt{R_2^2 + R_2^2 R_3^2 - 2R_2^2 R_3\cos\left(\theta_t - \theta_1\right)}}$$

If we determine the value of θ_1 which makes this derivative zero, we have found the optimum plane change required at the first impulse. Once θ_1 is calculated we can determine θ_2 from the total plane change angle relationship and the velocity impulses from the previous equations.

Numerical Solution

This numerical algorithm has been implemented in an interactive MATLAB script called `hohmann.m`. This script prompts the user for the initial and final altitudes in kilometers and the initial and final orbital inclinations in degrees. The software then calls the Brent root-finding algorithm to solve the partial derivative equation described above.

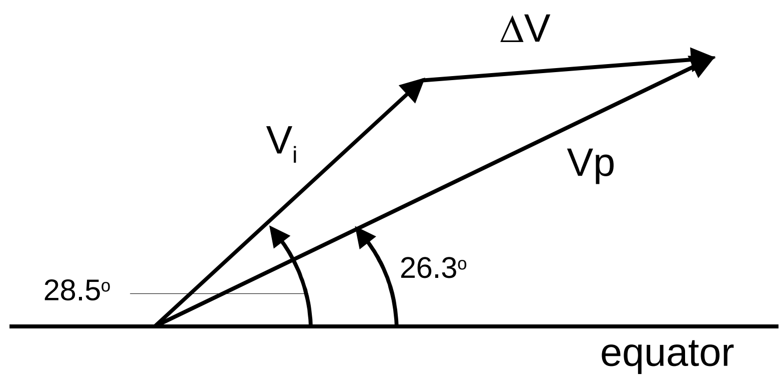
The call to the Brent root-finding algorithm is as follows:

```
[xroot, froot] = brent('hohmfunc', 0, dinc, rtol);
```

where `hohmfunc` is the objective function for this problem. Since we know that the optimum first plane change angle is somewhere between `0` and the total plane change angle `dinc`, we pass these as the bounds of the root. In the parameter list `rtol` is the user-defined root-finding convergence tolerance.

The following is a typical orbit transfer from a low altitude Earth orbit (LEO) at an altitude of 185.2 kilometers and an orbital inclination of 28.5 degrees to a geosynchronous Earth orbit (GSO) at an altitude of 35786.36 kilometers and 0 degrees inclination.

The following is a ΔV diagram for the first maneuver of this orbit transfer example. In this view we are looking along the line of nodes which is the mutual intersection of the park and transfer orbit planes with the equatorial plane.



In this diagram V_i is the speed on the initial park orbit, V_p is the perigee speed of the elliptic transfer orbit, and ΔV is the impulse required for the first maneuver. The inclinations of the park and transfer orbit are also labeled. From this geometry and the law of cosines, the required ΔV is given by

$$\Delta V = \sqrt{V_i^2 + V_p^2 - 2 V_i V_p \cos \Delta i}$$

where Δi is the inclination difference or plane change between the park and transfer orbits.

User interaction with the script

The following is a typical user interaction with this MATLAB script. User inputs are in bold font.

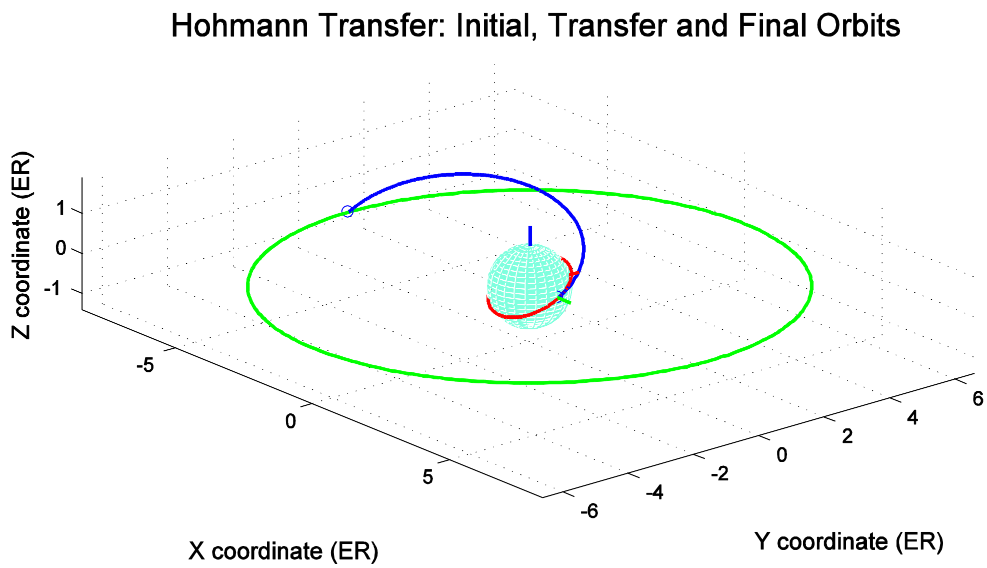
```
Hohmann Orbit Transfer Analysis


please input the initial altitude kilometers
? 300

please input the final altitude kilometers
? 35786.2

please input the initial orbital inclination degrees
(0 <= inclination <= 180)
? 28.5

please input the final orbital inclination degrees
(0 <= inclination <= 180)
? 0
```

The following is the script solution for this example.

```
Hohmann Orbit Transfer Analysis
-------------------------------

initial orbit altitude             300.0000 kilometers

initial orbit radius              6678.1363 kilometers

initial orbit inclination           28.5000 degrees

initial orbit velocity            7725.7606 meters/second


final orbit altitude             35786.2000 kilometers

final orbit radius               42164.3363 kilometers

final orbit inclination              0.0000 degrees

final orbit velocity              3074.6540 meters/second


first inclination change             2.2002 degrees

second inclination change           26.2998 degrees

total inclination change            28.5000 degrees


first delta-v                     2449.4551 meters/second

second delta-v                    1781.8532 meters/second

total delta-v                     4231.3083 meters/second


transfer orbit semimajor axis    24421.2363 kilometers

transfer orbit eccentricity      0.72654389

transfer orbit inclination          26.2998 degrees

transfer orbit perigee velocity  10151.4962 meters/second

transfer orbit apogee velocity    1607.8298 meters/second

transfer orbit coast time        18990.3276 seconds
                                   316.5055 minutes
                                     5.2751 hours
```

This MATLAB script is valid for Hohmann transfers from a high initial circular orbit to a lower final orbit. It also handles the case of transfer to a mission orbit with higher orbital inclination.

The `hohmann` script will also create a graphics display of the initial, transfer and final orbits. The following is the graphics display for this example. The initial orbit trace is red, the transfer orbit is blue and the final mission orbit is green. The dimensions are Earth radii (ER) and the plot is labeled with an ECI coordinate system where green is the x-axis, red is the y-axis and blue is the z-axis. The location of each impulse is marked with a small blue circle.



The interactive graphic features of MATLAB allow the user to rotate and zoom the display. These capabilities allow the user to interactively find the best viewpoint as well as verify basic three-dimensional geometry of the orbital transfer.

The `hohmann` MATLAB script will also create color a Postscript disk file of this graphic image. This image includes a TIFF preview and is created with MATLAB code similar to

```
print -depsc -tiff -r300 hohmann1.eps
```

Primer Vector Analysis

This section summarizes the primer vector analysis included with this MATLAB script. The term primer vector was invented by Derek F. Lawden and represents the adjoint vector for velocity. A technical discussion about primer theory can be found in Lawden's classic text, *Optimal Trajectories for Space Navigation*, Butterworths, London, 1963. Another excellent resource is "Primer Vector Theory and Applications", Donald J. Jezewski, NASA TR R-454, November 1975, along with "Optimal, Multi-burn, Space Trajectories", also by Jezewski.

As shown by Lawden, the following four necessary conditions must be satisfied in order for an impulsive orbital transfer to be *locally optimal*:

(1) the primer vector and its first derivative are everywhere continuous

(2) whenever a velocity impulse occurs, the primer is a unit vector aligned with the impulse and has unit magnitude $\left(\mathbf{p} = \hat{\mathbf{p}} = \hat{\mathbf{u}}_T \text{ and } \|\mathbf{p}\| = 1\right)$

(3) the magnitude of the primer vector may not exceed unity on a coasting arc $\left(\|\mathbf{p}\| = p \le 1\right)$

(4) at all interior impulses (not at the initial or final times) $\mathbf{p} \bullet \dot{\mathbf{p}} = 0$; therefore, $d\|\mathbf{p}\|/dt = 0$ at the intermediate impulses

Furthermore, the scalar magnitudes of the primer vector derivative at the initial and final impulses provide information about how to improve the nominal transfer trajectory by changing the endpoint times and/or moving the impulse times. These four cases for non-zero slopes are summarized as follows;

- If $\dot{p}_0 > 0$ and $\dot{p}_f < 0 \rightarrow$ perform an initial coast before the first impulse and add a final coast after the second impulse
- If $\dot{p}_0 > 0$ and $\dot{p}_f > 0 \rightarrow$ perform an initial coast before the first impulse and move the second impulse to a later time
- If $\dot{p}_0 < 0$ and $\dot{p}_f < 0 \rightarrow$ perform the first impulse at an earlier time and add a final coast after the second impulse
- If $\dot{p}_0 < 0$ and $\dot{p}_f > 0 \rightarrow$ perform the first impulse at an earlier time and move the second impulse to a later time

The primer vector analysis of a two impulse orbital transfer involves the following steps.

First partition the two-body state transition matrix as follows:

$$\Phi\left(t,t_0\right) = \begin{bmatrix} \dfrac{\partial \mathbf{r}}{\partial \mathbf{r}_0} & \dfrac{\partial \mathbf{r}}{\partial \mathbf{v}_0} \\ \dfrac{\partial \mathbf{v}}{\partial \mathbf{r}_0} & \dfrac{\partial \mathbf{v}}{\partial \mathbf{v}_0} \end{bmatrix} = \begin{bmatrix} \Phi_{11} & \Phi_{12} \\ \Phi_{21} & \Phi_{22} \end{bmatrix} = \begin{bmatrix} \Phi_{rr} & \Phi_{rv} \\ \Phi_{vr} & \Phi_{vv} \end{bmatrix}$$

where

$$\Phi_{11} = \left[\frac{\partial \mathbf{r}}{\partial \mathbf{r}_0} \right] = \begin{bmatrix} \partial x / \partial x_0 & \partial x / \partial y_0 & \partial x / \partial z_0 \\ \partial y / \partial x_0 & \partial y / \partial y_0 & \partial y / \partial z_0 \\ \partial z / \partial x_0 & \partial z / \partial y_0 & \partial z / \partial z_0 \end{bmatrix}$$

and so forth.

The value of the primer vector at any time *t* along a two body trajectory is given by

$$\mathbf{p}\left(t\right) = \Phi_{11}\left(t,t_0\right)\mathbf{p}_0 + \Phi_{12}\left(t,t_0\right)\dot{\mathbf{p}}_0$$

and the value of the primer vector derivative is

$$\dot{\mathbf{p}}\left(t\right) = \Phi_{21}\left(t,t_0\right)\mathbf{p}_0 + \Phi_{22}\left(t,t_0\right)\dot{\mathbf{p}}_0$$

which can also be expressed as

$$\begin{Bmatrix} \mathbf{p} \\ \dot{\mathbf{p}} \end{Bmatrix} = \Phi\left(t, t_0\right) \begin{Bmatrix} \mathbf{p}_0 \\ \dot{\mathbf{p}}_0 \end{Bmatrix}$$

The primer vector boundary conditions at the initial and final impulses are as follows:

$$\mathbf{p}\left(t_0\right) = \mathbf{p}_0 = \frac{\Delta \mathbf{V}_0}{\left|\Delta \mathbf{V}_0\right|} \qquad \mathbf{p}\left(t_f\right) = \mathbf{p}_f = \frac{\Delta \mathbf{V}_f}{\left|\Delta \mathbf{V}_f\right|}$$

These two conditions illustrate that at the locations of velocity impulses, the primer vector is a unit vector in the direction of the corresponding impulse.

The value of the primer vector derivative at the initial time is

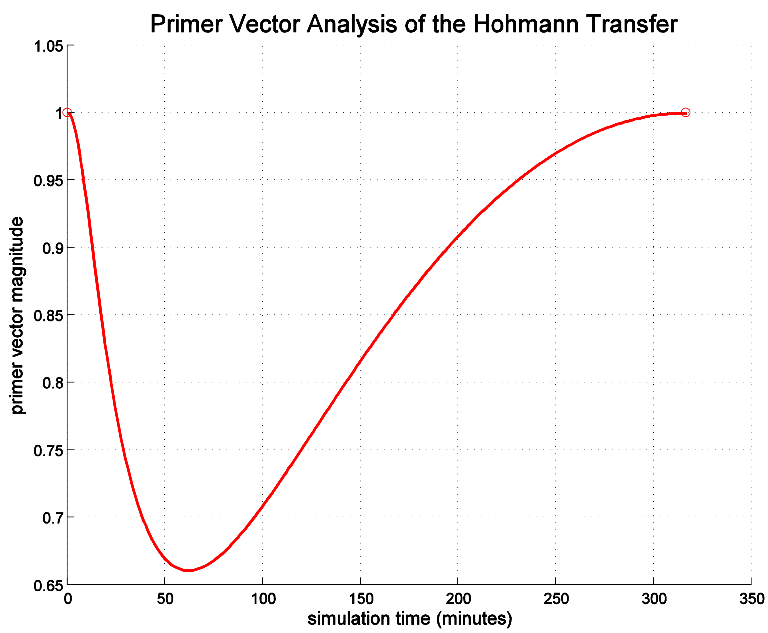
$$\dot{\mathbf{p}}\left(t_0\right) = \dot{\mathbf{p}}_0 = \Phi_{12}^{-1}\left(t_f, t_0\right)\left\{\mathbf{p}_f - \Phi_{11}\left(t_f, t_0\right)\mathbf{p}_0\right\}$$

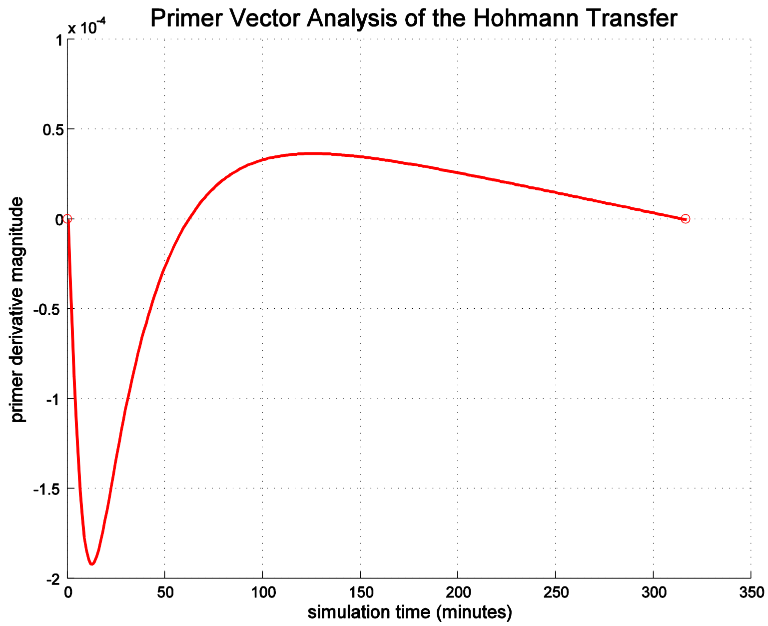
provided the Φ_{12} sub-matrix is non-singular.

The scalar magnitude of the derivative of the primer vector can be determined from

$$\frac{d\left\|\mathbf{p}\right\|}{dt} = \frac{d}{dt}\left(\mathbf{p} \bullet \mathbf{p}\right)^{2} = \frac{\dot{\mathbf{p}} \bullet \mathbf{p}}{\left\|\mathbf{p}\right\|}$$

The following two graphic images illustrate the behavior of the magnitudes of the primer vector and its derivative for the example given earlier. The location of each impulse is marked with a small red circle.





From the properties of the primer vector and its derivative, we can see that this orbit transfer is optimal.

The `hohmann` MATLAB script will also create color a Postscript disk file of these graphic images. This image includes a TIFF preview and is created with MATLAB source code similar to

```
print -depsc -tiff -r300 primer.eps
```

Algorithm resources

(1) Walter Hohmann, *Die Erreichbarkeit der Himmelskorper*, Oldenbourgh, Munich, 1925. Also, *The Attainability of Heavenly Bodies,* NASA Technical Translation F-44, 1960.

(2) Jean-Pierre Marec, *Optimal Space Trajectories*, Elsevier, 1979.

(3) R. P. Brent, *Algorithms for Minimization Without Derivatives*, Prentice-Hall, 1972.

(4) R. H. Battin, *An Introduction to the Mathematics and Methods of Astrodynamics*, AIAA, 1987.

(5) D. F. Lawden, *Optimal Trajectories for Space Navigation*, Butterworths, London, 1963.

(6) John E. Prussing, “Simple Proof of the Global Optimality of the Hohmann Transfer”, *AIAA Journal of Guidance, Control and Dynamics*, Vol. 15, No. 4.

(7) A. Miele, M. Ciarcia, and J. Mathwig, “Reflections on the Hohmann Transfer”, *Journal of Optimization Theory and Applications*, Vol. 123, No. 2, pp. 233-253, November 2004.